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Bachelor of Commerce (B2001) | Level 2 • Part B Major | Monash University

Intermediate microeconomics

Course Code: ECC2000 • Module 2 Notes

Module 2: Choice Under Uncertainty, Production Technology & Cost Minimisation

Chapter 5: Choice Under Uncertainty and Expected Utility Theory

5.1 Lotteries, States of Nature, and von Neumann–Morgenstern Axioms

Real-world economic decisions—from an Australian household choosing between fixed and variable mortgage rates benchmarked against the Reserve Bank of Australia (RBA) cash rate, to a Victorian grain grower purchasing hail insurance—occur under uncertainty. Following von Neumann and Morgenstern (1944), uncertainty is formalised via a finite set of mutually exclusive states of nature $s \in \{1, 2, \dots, S\}$ occurring with objective probabilities $\pi_1, \pi_2, \dots, \pi_S$ where $\pi_s \geq 0$ and $\sum_{s=1}^{S} \pi_s = 1$. A **simple lottery** $L = (c_1, \dots, c_S; \pi_1, \dots, \pi_S)$ specifies state-contingent wealth or consumption payouts c_s.

To represent preferences $\succsim$ over the space of lotteries L by an expected utility functional, four rationality axioms are imposed:

1. **Completeness and Transitivity:** All lotteries can be ranked consistently without cycles.
2. **Continuity (Archimedean Axiom):** If $L_1 \succ L_2 \succ L_3$, there exists a mixing probability $p \in (0, 1)$ such that the compound lottery $pL_1 + (1-p)L_3$ is indifferent to L_2.
3. **Independence Axiom:** For any lotteries L_1, L_2, L_3 and mixing weight $\alpha \in (0, 1]$, $L_1 \succsim L_2$ holds if and only if $\alpha L_1 + (1-\alpha)L_3 \succsim \alpha L_2 + (1-\alpha)L_3$.

The **Expected Utility Theorem** proves that preferences satisfy these axioms if and only if there exists a Bernoulli utility function $u : \mathrm{R}_{+} \to \mathrm{R}$ over certain wealth outcomes such that the von Neumann–Morgenstern (vNM) expected utility function ranks lotteries by:

$$U(L) = \mathrm{E}[u(W)] = \sum_{s=1}^{S} \pi_s u(c_s)$$

Where:

- $U(L)$ denotes the expected utility assigned to lottery L.
- π_s is the probability of state s, with $\sum_{s=1}^{S} \pi_s = 1$.
- $u(c_s)$ is the **Bernoulli (felicity) utility function** evaluated at state-contingent wealth c_s.

Unlike deterministic ordinal utility (which is invariant under *any* strictly increasing transformation), a vNM Bernoulli utility function $u($
$cdot)$ is cardinal up to **positive affine transformations**: $v(c) = a + bu(c)$ with $b > 0$ represents identical lottery rankings, because $\mathrm{E}[a + bu(W)] = a + b\mathrm{E}[u(W)]$. Non-linear transformations (such as squaring or taking logarithms of u) distort risk attitudes and alter lottery preferences. Empirical violations of the Independence Axiom—most famously the **Allais Paradox**, where individuals exhibit disproportionate preference for certainty when comparing sure outcomes against near-certain gambles—motivate behavioural alternatives such as Prospect Theory, though vNM expected utility remains the benchmark for rational insurance and finance theory.

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5.2 Risk Attitudes, Jensen's Inequality, and Certainty Equivalents

Consider a gamble yielding random final wealth W with expected wealth $\mathrm{E}[W] = \sum_{s=1}^{S} \pi_s c_s$. Comparing the utility of the expected payout $u(\mathrm{E}[W])$ against the expected utility of the lottery $\mathrm{E}[u(W)]$ classifies an agent's attitude toward risk:

- **Risk Averse:** Strictly prefers a sure wealth of $\mathrm{E}[W]$ to the risky lottery W ($u(\mathrm{E}[W]) > \mathrm{E}[u(W)]$). By **Jensen's Inequality**, this holds for all non-degenerate lotteries if and only if $u(W)$ is strictly concave ($u''(W) < 0$).
- **Risk Neutral:** Indifferent between a sure payment of $\mathrm{E}[W]$ and the risky lottery ($u(\mathrm{E}[W]) = \mathrm{E}[u(W)]$), corresponding to a linear Bernoulli function ($u''(W) = 0$).
- **Risk Loving:** Strictly prefers the risky lottery to its expected value ($u(\mathrm{E}[W]) < \mathrm{E}[u(W)]$), corresponding to a strictly convex Bernoulli function ($u''(W) > 0$).

Two monetary metrics quantify the intensity of risk aversion for any lottery W:

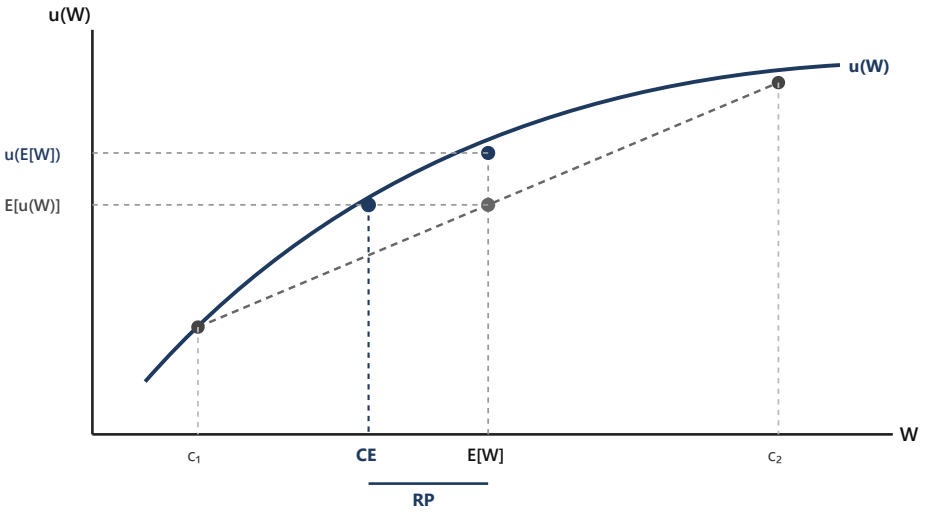
- **Certainty Equivalent (CE):** The guaranteed dollar wealth that yields the same utility as the risky lottery:

  $$u(CE) = \mathrm{E}[u(W)]$$
  $$implies CE = u^{-1}(\mathrm{E}[u(W)])$$

  Where u^{-1} is the inverse Bernoulli utility function. For a risk-averse agent, $CE < \mathrm{E}[W]$.
- **Risk Premium (RP):** The maximum dollar amount a decision-maker is willing to forfeit from expected wealth to eliminate all risk:

  $$RP = \mathrm{E}[W] - CE$$

  Where $RP > 0$ for risk aversion, $RP = 0$ for risk neutrality, and $RP < 0$ for risk loving.



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5.3 Arrow-Pratt Measures of Risk Aversion

Simply inspecting the second derivative $u''(W)$ is unsuitable as a measure of risk aversion because multiplying $u(W)$ by a positive constant b scales $u''(W)$ by b without altering choice behaviour. Kenneth Arrow and John Pratt independently constructed scale-invariant local measures of curvature:

$$ARA(W) = -\frac{u''(W)}{u'(W)},$$
$$qquadRRA(W) = -\frac{Wu''(W)}{u'(W)} = W$$
$$cdotARA(W)$$

Where:

- $ARA(W)$ is the **Arrow-Pratt coefficient of Absolute Risk Aversion** (units: dollars^{-1}), governing willingness to take a fixed-dollar gamble $\tilde{\epsilon}$ as wealth W rises.
- $RRA(W)$ is the **Arrow-Pratt coefficient of Relative Risk Aversion** (dimensionless), governing willingness to invest a fixed *fraction* of wealth in a risky asset as W rises.

Taking a second-order Taylor expansion of $\mathrm{E}[u(W_0 + \tilde{\epsilon})]$ around initial wealth W_0 for a small zero-mean risk $\tilde{\epsilon}$ with variance σ^2 and equating it to $u(W_0 - RP) \approx u(W_0) - RP$ $cdotu'(W_0)$ yields **Pratt's local risk-premium approximation**:

$$RP \approx \frac{1}{2}\sigma^2$$
$$left(-\frac{u''(W_0)}{u'(W_0)}$$
$$right) = \frac{1}{2}\sigma^2 ARA(W_0)$$

Where $\sigma^2 = \mathrm{E}[\tilde{\epsilon}^2]$ is the variance of the zero-mean wealth shock. Three standard utility families illustrate how ARA and RRA vary with wealth:

Utility Family	Functional Form	$ARA(W)$	$RRA(W)$	Wealth Behaviour
CARA (Exponential)	$u(W) = -e^{-aW}$ ($a > 0$)	a (constant)	aW (increasing)	Constant dollar amount invested in risky assets regardless of wealth.
CRRA (Power)	$u(W) = \frac{W^{1-\gamma}}{1-\gamma}$ ($\gamma > 0, \gamma \neq 1$)	$\frac{\gamma}{W}$ (decreasing)	γ (constant)	Constant portfolio share invested in risky assets (DARA, CRRA).
Logarithmic	$u(W) = \ln W$ ($\gamma = 1$ limit)	$\frac{1}{W}$ (decreasing)	1 (constant)	Special case of CRRA with $\gamma = 1$; exhibits DARA.

5.4 First-Order and Second-Order Stochastic Dominance

When comparing two risky return distributions $F(w)$ and $G(w)$ over support $[a, b]$ without knowing an investor's exact Bernoulli utility function, microeconomics relies on **stochastic dominance** criteria:

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- **First-Order Stochastic Dominance (FOSD):** Distribution F first-order stochastically dominates G if $F(w) \leq G(w)$ for all $w \in [a, b]$ (meaning F always places at least as much cumulative probability mass on higher wealth outcomes). Every expected-utility maximiser with a non-decreasing utility function ($u' \geq 0$) weakly prefers F to G.
- **Second-Order Stochastic Dominance (SOSD):** When two distributions have the same mean ($\mathrm{E}_F[W] = \mathrm{E}_G[W]$), F second-order stochastically dominates G if $\int_a^x [G(w) - F(w)]\, dw \geq 0$ for all $x \in [a, b]$ (equivalently, G is formed from F by adding a zero-mean noise variable, a **mean-preserving spread**). Every risk-averse decision-maker ($u' \geq 0, u'' \leq 0$) weakly prefers F to G.

Worked Example 5.1: Certainty Equivalent, Risk Premium, and Arrow-Pratt Measures

A Melbourne investor holds initial wealth W_0 = \$100,000 and evaluates an ASX technology venture offering an equal 50% chance of final wealth \$160,000 and a 50% chance of final wealth \$64,000. Compare the investor's evaluation under logarithmic utility $u(W) = \ln W$ versus CRRA utility with $\gamma = 2$, $v(W) = -W^{-1}$.

1. **Step 1 — Expected wealth:**

$$\mathrm{E}[W] = 0.5(160,000) + 0.5(64,000) = 112,000$$

   Where $\mathrm{E}[W]$ equals \$112,000.

2. **Step 2 — Logarithmic utility ($\gamma = 1$):**

$$\mathrm{E}[\ln W] = 0.5 \ln(160,000) + 0.5 \ln(64,000) = \ln(sqrt160,000 \times 64,000) = \ln(101,192.89)$$

   Setting $\ln(CE_{\ln}) = \ln(101,192.89)$ gives $CE_{\ln}$ = \$101,192.89 and $RP_{\ln} = 112,000 - 101,192.89$ = \$10,807.11. Here $ARA(W) = 1/W$ and $RRA(W) = 1$.

3. **Step 3 — CRRA utility with $\gamma = 2$:** With $v'(W) = W^{-2}$ and $v''(W) = -2W^{-3}$, we verify $ARA(W) = 2/W$ and $RRA(W) = 2$. Expected utility is:

$$\mathrm{E}[v(W)] = 0.5 left(-\frac{1}{160,000} right) + 0.5 left(-\frac{1}{64,000} right) = -0.000003125 - 0.0000078125 = -0.0000109375$$

   Setting $-CE_2^{-1} = -0.0000109375$ gives $CE_2 = 1/0.0000109375$ = \$91,428.57, and risk premium $RP_2 = 112,000 - 91,428.57$ = \$20,571.43. Because $\gamma = 2$ exhibits higher risk aversion than $\gamma = 1$, the investor accepts a lower certainty equivalent (\$91,428.57 vs \$101,192.89) and rejects the gamble relative to holding \$100,000 safely.

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Chapter 6: Insurance Markets, Risk Sharing, and Portfolio Choice

6.1 State-Space Analysis and the Full Insurance Theorem

Suppose a Sydney café owner holds initial wealth W_0 and faces a probability $pi \in (0,1)$ of a fire loss $D > 0$ (state 1: loss) and probability $1 - pi$ of no loss (state 2: no loss). The owner can purchase K dollars of insurance coverage at a unit premium rate $\gamma \in (0,1)$ per dollar of coverage, paying premium γK in both states. State-contingent consumption levels are:

$$c_1 = W_0 - D + K - \gamma K = W_0 - D + (1-\gamma)K, \\ qquadc_2 = W_0 - \gamma K$$

Where:

- c_1 and c_2 denote final wealth in the loss state and the no-loss state, respectively.
- $K \geq 0$ is the insured payout (indemnity) chosen by the policyholder.
- γ is the premium per dollar of coverage charged by the insurer.

Eliminating K between the two equations gives the **state-contingent budget constraint** passing through the endowment point $(W_0 - D, W_0)$ with slope $-\frac{\gamma}{1-\gamma}$:

$$\gamma c_1 + (1-\gamma)c_2 = W_0 - \gamma D$$

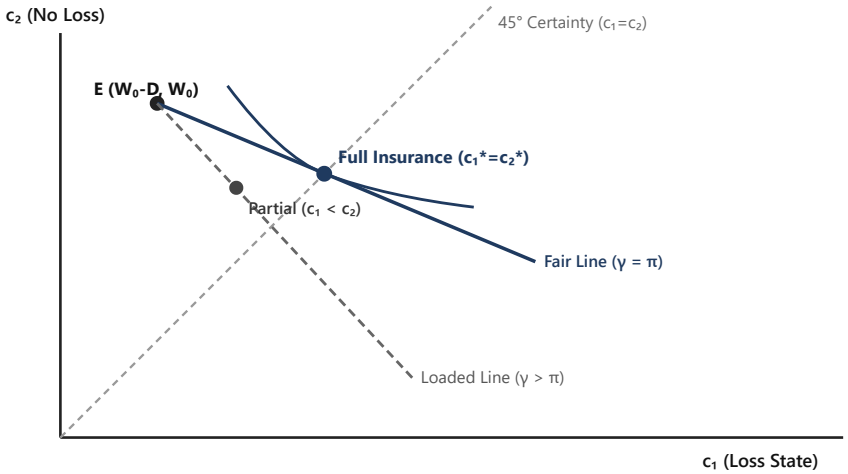
The café owner maximises expected utility $pi u(c_1) + (1 - pi)u(c_2)$ subject to this constraint. Setting the marginal rate of substitution between state-1 and state-2 wealth equal to the price ratio gives the interior optimum:

$$MRS_{1,2} = \frac{pi u'(c_1)}{(1-pi)u'(c_2)} = \frac{\gamma}{1-\gamma}$$

Where:

- $pi/(1 - pi)$ is the objective odds ratio of the loss state.
- $\gamma/(1-\gamma)$ is the market odds price of shifting one dollar into the loss state.

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An insurance policy is **actuarially fair** when the expected profit of the insurer is zero: $\gamma K - piK = 0 \iff \gamma = pi$. Substituting $\gamma = pi$ into the optimality condition yields $u^{'}(c_1^*) = u^{'}(c_2^*)$. For any strictly risk-averse individual ($u^{''} < 0$, so $u^{'}$ is strictly decreasing), this \implies $c_1^* = c_2^*$: the consumer buys **full insurance** ($K^* = D$), completely eliminating risk along the 45° certainty line. Conversely, when administrative expenses or market power produce an **actuarially unfair (loaded)** premium with $\gamma > pi$, optimality requires $u^{'}(c_1^*) > u^{'}(c_2^*)$, which by concavity \implies $c_1^* < c_2^*$ ($K^* < D$): the risk-averse agent rationally buys **partial insurance**, retaining a deductible or co-payment.

6.2 Portfolio Choice Between Safe and Risky Assets

Consider an investor with initial wealth W_0 allocating dollar amount $a \in [0, W_0]$ to a risky asset (such as an S&P/ASX 200 index fund) with random gross return $1 + \tilde{r}$ and the remainder $W_0 - a$ to a risk-free Australian Government Bond with certain gross return $1 + r_f$. Final random wealth is:

$$\tilde{W}(a) = (W_0 - a)(1 + r_f) + a(1 + \tilde{r}) = W_0(1 + r_f) + a(\tilde{r} - r_f)$$

Where:

- a is the dollar allocation to the risky asset.
- $\tilde{r} - r_f$ is the random excess return (equity risk premium).

Maximising $\mathrm{E}[u(\tilde{W}(a))]$ with respect to a gives the first-order condition $\mathrm{E}[u^{'}(\tilde{W}(a))(\tilde{r} - r_f)] = 0$. Evaluating this derivative at $a = 0$ (zero risky investment, where $\tilde{W}(0) = W_0(1 + r_f)$ is non-random) yields $u^{'}(W_0(1 + r_f)) cdot \mathrm{E}[\tilde{r} - r_f]$. Whenever the risky asset offers a strictly positive expected excess return ($\mathrm{E}[\tilde{r}] > r_f$), this marginal utility derivative at $a = 0$ is strictly positive. Hence, **any smooth risk-averse investor holds a strictly positive share of the risky asset** ($a^* > 0$), because a small risky bet incurs only a second-order variance cost while capturing a first-order mean return gain. Moreover, under DARA ($ARA^{'}(W) < 0$), the optimal dollar investment a^* strictly increases with initial wealth W_0.

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Worked Example 6.1: Optimal Insurance Coverage with Fair vs Loaded Premiums

A Brisbane homeowner with logarithmic utility $u(c) = \ln c$ has initial wealth W_0 = \$400,000 and faces a 10% probability ($pi = 0.10$) of cyclone damage costing D = \$190,000.

1. **Step 1 — Actuarially fair insurance ($\gamma = pi = 0.10$):** Since $\gamma = pi$, the optimality condition $u'(c_1) = u'(c_2)$ dictates full insurance: $K^* = D$ = \$190,000, costing premium $0.10 \times 190,000$ = \$19,000. Final wealth in both states is $c_1^* = c_2^* = 400,000 - 19,000$ = \$381,000.
2. **Step 2 — Heavily loaded insurance ($\gamma = 0.20 > pi = 0.10$):** Using $\frac{pi/c_1}{(1-pi)/c_2} = \frac{\gamma}{1-\gamma}$:

$$\frac{0.10c_2}{0.90c_1} = \frac{0.20}{0.80} = \frac{1}{4}$$
$$implies c_2 = \frac{9}{4}c_1 = 2.25c_1$$

   Substituting $c_1 = 210,000 + 0.8K$ and $c_2 = 400,000 - 0.2K$:

$$400,000 - 0.2K = 2.25(210,000 + 0.8K) = 472,500 + 1.8K$$

   Notice $400,000 - 472,500 = 2.0K$ gives $K < 0$. Because coverage is constrained to $K \geq 0$, the homeowner hits the boundary $K^* = 0$ and self-insures at $\gamma = 0.20$.
3. **Step 3 — Moderate loading ($\gamma = 0.12$):** Now $\frac{0.10c_2}{0.90c_1} = \frac{0.12}{0.88} = \frac{3}{22}$ $implies c_2 = \frac{27}{22}c_1 \approx 1.2273c_1$. Substituting $c_1 = 210,000 + 0.88K$ and $c_2 = 400,000 - 0.12K$:

$$400,000 - 0.12K = 1.2273(210,000 + 0.88K) = 257,727.27 + 1.08K$$

   Solving $1.20K = 142,272.73$ yields K^* = \$118,560.61 (partial insurance with a self-retained deductible of \$71,439.39).

Chapter 7: Production Technology, Isoquants, MRTS, and Returns to Scale

7.1 Production Functions, Short-Run Productivity, and Isoquants

A firm converts inputs—typically capital (K, machine-hours) and labour (L, worker-hours)—into final output (q) via a technology represented by a **production function** $q = f(K, L)$, defined as the maximum technically feasible output attainable from (K, L). In the **short run**, at least one input is fixed (customarily $K = \bar{K}$), whereas in the **long run** all inputs are freely variable.

Holding capital fixed at $\bar{K}$, short-run labour productivity is described by:

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- **Marginal Product of Labour (MP_L):** $MP_L = \frac{partialf(K,L)}{partialL}$ (extra output from an additional unit of labour).
- **Average Product of Labour (AP_L):** $AP_L = \frac{f(\bar{K},L)}{L}$ (output per worker-hour).

The **Law of Diminishing Marginal Returns** states that as successive units of a variable input are added to fixed inputs, MP_L eventually declines ($\frac{partial^2f}{partialL^2} < 0$). Furthermore, differentiating $AP_L = q/L$ gives $\frac{dAP_L}{dL} = \frac{1}{L}(MP_L - AP_L)$: whenever $MP_L > AP_L$, AP_L rises; whenever $MP_L < AP_L$, AP_L falls; thus MP_L intersects AP_L at the exact maximum of AP_L.

In the long run, an **isoquant** is the locus of all input bundles (L,K) capable of producing a fixed output level $\bar{q}$: $Q(\bar{q}) = \{(L,K) \in \mathrm{R}_+^2 : f(K,L) = \bar{q}\}$. Unlike utility, output q is cardinally measurable (in tonnes, megawatt-hours, or units). The absolute slope of an isoquant is the **Marginal Rate of Technical Substitution ($MRTS_{L,K}$)**, measuring how much capital can be released when one extra unit of labour is hired while holding output fixed at $\bar{q}$:

$$df = MP_L\,dL + MP_K\,dK = 0$$
$$impliesMRTS_{L,K} = -\frac{dK}{dL}\bigg|_{q=\bar{q}} = \frac{MP_L(K,L)}{MP_K(K,L)}$$

Where:

- $MP_L =$ $partialf/$ $partialL$ and $MP_K =$ $partialf/$ $partialK$ are the marginal products of labour and capital.
- $MRTS_{L,K}$ decreases as L rises and K falls whenever isoquants are convex (quasiconcave production function).

7.2 Elasticity of Factor Substitution, Returns to Scale, and Euler's Theorem

The ease with which a firm substitutes labour for capital along an isoquant is measured by the **elasticity of technical substitution (σ)**:

$$\sigma = \frac{d\ln(K/L)}{d\ln(MRTS_{L,K})}$$

Where K/L is the capital-labour ratio. For a Cobb-Douglas technology $q = AK^{\alpha}L^{\beta}$, $MRTS_{L,K} = \frac{\beta}{\alpha}\frac{K}{L}$, so $\sigma = 1$ identically. For Leontief fixed-proportions technology $q = \min\{aK, bL\}$, $\sigma = 0$; for linear perfect-substitutes inputs $q = aK + bL$, $\sigma = \infty$; and for CES production $q = A(\alpha K^{\rho} + \beta L^{\rho})^{v/\rho}$, $\sigma = \frac{1}{1-\rho}$.

While diminishing marginal returns is a *short-run* property of varying one input, **Returns to Scale (RTS)** is a *long-run* property describing how output responds when *all* inputs are scaled proportionally by a scalar $t > 1$:

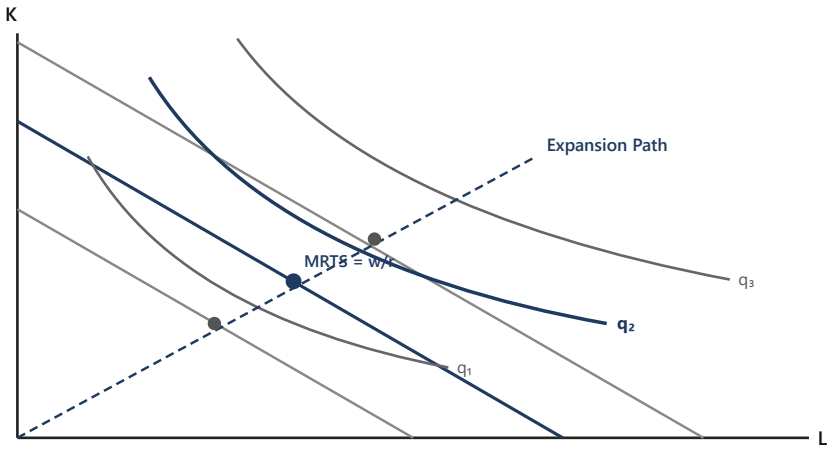
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- **Increasing Returns to Scale (IRS):** $f(tK, tL) > tf(K, L)$ (doubling all inputs more than doubles output, often due to specialisation or geometric container/pipeline scaling).
- **Constant Returns to Scale (CRS):** $f(tK, tL) = tf(K, L)$ (output scales strictly proportionally; homogeneous of degree 1).
- **Decreasing Returns to Scale (DRS):** $f(tK, tL) < tf(K, L)$ (doubling inputs less than doubles output, often due to managerial coordination bottlenecks).

More generally, if $f(tK, tL) = t^k f(K, L)$ (homogeneous of degree k), the **elasticity of scale** equals $k = \varepsilon_L + \varepsilon_K$, the sum of output elasticities ($\varepsilon_L = \frac{partialf}{partialL}\frac{L}{q}$, $\varepsilon_K = \frac{partialf}{partialK}\frac{K}{q}$). By **Euler's Homogeneous Function Theorem**:

$$
\begin{gathered}
L \\
cdotMP_L + K \\
cdotMP_K = k \\
cdotf(K, L)
\end{gathered}
$$

Where k is the degree of homogeneity. Under Constant Returns to Scale ($k = 1$) and competitive factor pricing ($w = p$
$cdotMP_L, r = p$
$cdotMP_K$), total factor payments satisfy $wL + rK = p(L$
$cdotMP_L + K$
$cdotMP_K) = pq$: paying every factor its marginal revenue product exactly exhausts total revenue, leaving zero economic profit (the **Product Exhaustion Theorem**).



Worked Example 7.1: Productivity, MRTS, Returns to Scale, and Euler Check

An advanced packaging plant in Geelong operates with production function $q = 10K^{1/4}L^{1/2}$.

1. **Step 1 — Marginal products and diminishing returns:**

$$MP_L = \frac{partialq}{partialL} = 5K^{1/4}L^{-1/2},$$
$$qquadMP_K = \frac{partialq}{partialK} = 2.5K^{-3/4}L^{1/2}$$

Since $\frac{partialMP_L}{partialL} = -2.5K^{1/4}L^{-3/2} < 0$ and $\frac{partialMP_K}{partialK} = -1.875K^{-7/4}L^{1/2} < 0$, both inputs exhibit diminishing marginal returns.

2. **Step 2 — Marginal Rate of Technical Substitution:**

$$MRTS_{L,K} = \frac{MP_L}{MP_K} = \frac{5K^{1/4}L^{-1/2}}{2.5K^{-3/4}L^{1/2}} = 2\frac{K}{L}$$

At $(K = 16, L = 8)$, $MRTS_{L,K} = 2(16/8) = 4$: the plant can replace 4 machine-hours with 1 extra worker-hour while keeping output constant.

3. **Step 3 — Returns to scale and Euler's Theorem verification:** Scaling inputs by $t > 1$ gives:

$$f(tK, tL) = 10(tK)^{1/4}(tL)^{1/2} = t^{3/4}$$
$$left(10K^{1/4}L^{1/2}$$
$$right) = t^{0.75}f(K, L)$$

Since $t^{0.75} < t$ for $t > 1$ (degree $k = 0.75 < 1$), the plant exhibits **decreasing returns to scale**. Checking Euler's Theorem: L
$cdotMP_L + K$
$cdotMP_K = 5K^{1/4}L^{1/2} + 2.5K^{1/4}L^{1/2} = 7.5K^{1/4}L^{1/2} = 0.75q.$

Chapter 8: Cost Minimisation, Conditional Factor Demands & Cost Functions

8.1 The Long-Run Cost Minimisation Problem and Shephard's Lemma

Given competitive input prices—wage rate $w > 0$ and rental rate of capital $r > 0$—a cost-minimising firm chooses (K, L) to produce a target output $q > 0$ at minimum total expenditure:

$$C(w, r, q) = \min_{K,L \geq 0} wL + rK \quad \text{subject to} \quad f(K, L) \geq q$$

Where:

- $C(w, r, q)$ is the long-run **total cost function**.
- w and r are unit factor prices for labour and capital, respectively.

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Forming the Lagrangian $\mathrm{L} = wL + rK + \lambda(q - f(K,L))$ yields the interior first-order conditions $\frac{partial\mathrm{L}}{partialL} = w - \lambda MP_L = 0$ and $\frac{partial\mathrm{L}}{partialK} = r - \lambda MP_K = 0$. Dividing the two equations gives the cost-minimising tangency rule:

$$MRTS_{L,K} = \frac{MP_L}{MP_K} = \frac{w}{r} \iff \frac{MP_L}{w} = \frac{MP_K}{r} = \frac{1}{\lambda}$$

Where $\lambda = \frac{partialC}{partialq} = LMC(q)$ equals the firm's long-run marginal cost. Economically, cost minimisation requires equating the marginal output per dollar spent on labour (MP_L/w) to the marginal output per dollar spent on capital (MP_K/r). If $MP_L/w > MP_K/r$, the firm lowers total cost for the same output by shifting expenditure from capital toward labour.

Solving the tangency condition together with $f(K,L) = q$ yields the **conditional factor demands** $L^*(w,r,q)$ and $K^*(w,r,q)$. Substituting them into $wL + rK$ gives the cost function $C(w,r,q) = wL^*(w,r,q) + rK^*(w,r,q)$. Analogous to the consumer expenditure function, $C(w,r,q)$ is homogeneous of degree 1 in (w,r), concave and non-decreasing in (w,r), and satisfies **Shephard's Lemma for production**:

$$L^*(w,r,q) = \frac{partialC(w,r,q)}{partialw},$$
$$qquadK^*(w,r,q) = \frac{partialC(w,r,q)}{partialr}$$

Where $L^*(w,r,q)$ and $K^*(w,r,q)$ are the cost-minimising input quantities required to produce output q. By concavity of $C(w,r,q)$ in (w,r), $\frac{partialL^*}{partialw} = \frac{partial^2C}{partialw^2} \leq 0$ (conditional factor demand curves always slope downward in own price), and cross-price effects are symmetric: $\frac{partialL^*}{partialr} = \frac{partialK^*}{partialw}$.

8.2 Short-Run vs Long-Run Cost Curves and the Envelope Theorem

In the short run with capital fixed at $K = \bar{K}$, total cost is $STC(q;\bar{K}) = r\bar{K} + wL_{SR}(q;\bar{K}) = FC + VC(q)$, where $FC = r\bar{K}$ is fixed cost and $VC(q)$ is variable cost. Dividing by q yields Short-Run Average Total Cost ($SAC = AFC + AVC$), and differentiating yields Short-Run Marginal Cost ($SMC = dSTC/dq = w/MP_L$). Whenever MP_L rises, SMC falls; when MP_L falls, SMC rises; and SMC intersects both AVC and SAC at their respective minimum points.

In the long run, the firm chooses capital optimally as $K = K^*(w,r,q)$. Consequently, $LTC(q) = STC(q;K^*(q)) \leq STC(q;\bar{K})$ for every q. The Long-Run Average Cost curve ($LAC = LTC(q)/q$) forms the **lower envelope** of all SAC curves: at the output q_0 where $K^*(q_0) = \bar{K}$, $LAC(q_0) = SAC(q_0;\bar{K})$ and the two curves are tangent. By the Envelope Theorem:

$$LMC(q) = \frac{dLTC(q)}{dq} = \frac{partialSTC(q;K^*(q))}{partialq} = SMC(q;K^*(q))$$

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Where LMC equals SMC at the exact output for which the fixed plant size $\bar{K}$ is long-run optimal. Furthermore, returns to scale govern the slope of LAC:

- **Increasing returns to scale (economies of scale):** $LTC(q)$ grows less than proportionally with q, so $LAC(q)$ is downward sloping and $LMC(q) < LAC(q)$.
- **Constant returns to scale:** $LTC(q) = c$ $cdotq$, so $LAC(q) = LMC(q) = c$ is horizontal.
- **Decreasing returns to scale (diseconomies of scale):** $LTC(q)$ grows faster than q, so $LAC(q)$ is upward sloping and $LMC(q) > LAC(q)$.

8.3 Multi-Plant Cost Minimisation

When a multi-plant firm—such as an Australian electricity generator operating two power stations or a national manufacturer running Melbourne and Sydney factories—produces total output $Q = q_1 + q_2$ across two plants with cost functions $C_1(q_1)$ and $C_2(q_2)$, minimising $C_1(q_1) + C_2(q_2)$ subject to $q_1 + q_2 = Q$ requires equating marginal costs across all active plants:

$$MC_1(q_1^*) = MC_2(q_2^*)$$

Where $q_1^* + q_2^* = Q$ (assuming interior production $q_1^*, q_2^* > 0$; if $MC_1(0) > MC_2(Q)$, the entire output is assigned to Plant 2). Intuitively, if $MC_1 < MC_2$, shifting one unit of production from Plant 2 to Plant 1 holds total output fixed while reducing total cost by $MC_2 - MC_1 > 0$.

Worked Example 8.1: Conditional Factor Demands, Cost Function, and Envelope Verification

Using the Geelong packaging plant technology $q = 10K^{1/4}L^{1/2}$ from Worked Example 7.1, suppose the hourly wage is w and the hourly capital rental rate is r.

1. **Step 1 — Tangency condition:** Setting $MRTS_{L,K} = w/r$ gives:

$$2\frac{K}{L} = \frac{w}{r}$$
$$impliesK = \frac{w}{2r}L$$

   Where $w/(2r)$ determines the slope of the linear expansion path in (L, K) space.

2. **Step 2 — Conditional factor demands:** Substitute $K = \frac{w}{2r}L$ into $q = 10K^{1/4}L^{1/2}$:

$$q = 10$$
$$left(\frac{w}{2r}$$
$$right)^{1/4}L^{1/4}L^{1/2} = 10$$
$$left(\frac{w}{2r}$$
$$right)^{1/4}L^{3/4}$$

   Solving for L and K gives the conditional factor demands:

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$$L^*(w,r,q) = left(\frac{q}{10}right)^{4/3} left(\frac{2r}{w}right)^{1/3}, qquad K^*(w,r,q) = \frac{w}{2r}L^* = left(\frac{q}{10}right)^{4/3} left(\frac{w}{2r}right)^{2/3}$$

3. **Step 3 — Long-run cost function and Shephard's Lemma check:**

$$C(w,r,q) = wL^* + rK^* = left(\frac{q}{10}right)^{4/3} w^{2/3} r^{1/3} left(2^{1/3} + 2^{-2/3}right) = \frac{3}{2^{2/3}} left(\frac{q}{10}right)^{4/3} w^{2/3} r^{1/3}$$

Differentiating $C(w,r,q)$ with respect to w yields $\frac{partialC}{partialw} = left(\frac{q}{10}right)^{4/3} left(\frac{2r}{w}right)^{1/3} = L^*$, verifying Shephard's Lemma. Notice also that $C(w,r,q)$ is proportional to $q^{4/3}$, so $LAC(q) \propto q^{1/3}$ is strictly upward sloping—consistent with decreasing returns to scale ($0.75 < 1$).

4. **Step 4 — Numerical comparison with Short-Run Cost ($\bar{K} = 16$, w = \$32, r = \$16):**
   - *Long-run cost at $q = 80$:* $w/(2r) = 32/32 = 1$, so $L^* = K^* = (80/10)^{4/3} = 8^{4/3} = 16$. Total long-run cost is $LTC(80) = 32(16) + 16(16)$ = \$768 (LAC = \$9.60 per unit).
   - *Short-run cost with $\bar{K} = 16$:* From $q = 10(16)^{1/4}L^{1/2} = 20L^{1/2}$, short-run labour requirement is $L_{SR} = (q/20)^2 = q^2/400$. Hence $STC(q) = 16(16) + 32(q^2/400) = 256 + 0.08q^2$.
   - *Envelope verification at $q = 80$:* $STC(80) = 256 + 0.08(6,400)$ = \$768 = $LTC(80)$, and $SMC(80) = 0.16(80)$ = \$12.80 per unit, matching $LMC(80) = \frac{4}{3}\frac{LTC(80)}{80} = \frac{4}{3}(9.60)$ = \$12.80 per unit. For any $q \neq 80$ (say $q = 100$), $STC(100) = 256 + 800$ = \$1,056 strictly exceeds $LTC(100) = 768 \times (100/80)^{4/3}$ = \$1,034.47, confirming the envelope property.

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Worked Example 8.2: Multi-Plant Dispatch in an Australian Power Generator

A Victorian energy firm must supply $Q = 120$ MWh using two plants with cost functions $C_1(q_1) = 400 + 20q_1 + 0.5q_1^2$ and $C_2(q_2) = 600 + 10q_2 + q_2^2$.

1. **Step 1 — Equate plant marginal costs:** $MC_1(q_1) = 20 + q_1$ and $MC_2(q_2) = 10 + 2q_2$. Setting $MC_1 = MC_2$ and using $q_2 = 120 - q_1$:

$$20 + q_1 = 10 + 2(120 - q_1) = 250 - 2q_1$$
$$implies 3q_1 = 230$$
$$implies q_1^* = 76.67 \text{ MWh}$$

   Hence $q_2^* = 120 - 76.67 = 43.33$ MWh, at common marginal cost $MC_1 = MC_2 = 96.67$ dollars per MWh.
2. **Step 2 — Minimum total cost:** $C_1(76.67) = 400 + 1,533.33 + 2,938.89$ = \$4,872.22, and $C_2(43.33) = 600 + 433.33 + 1,877.78$ = \$2,911.11, giving minimum combined cost of \$7,783.33. Notice fixed costs (\$400 and \$600) do not affect optimal short-run dispatch between active plants.

8.4 Duality in Production, Profit Maximisation & Factor Demands

A central insight of advanced neoclassical microeconomics is the principle of **duality** between technological representations (production functions), cost representations (cost functions), and economic objective functions (profit functions). While the cost minimisation problem (CMP) identifies the input combination that achieves a predetermined target output at minimum expenditure, the unconstrained **Profit Maximisation Problem (PMP)** endogenises both input choices and the optimal output scale:

$$\max_{K,L \geq 0} \pi(p, w, r) = p \cdot f(K, L) - wL - rK$$

Where:

$\pi(p, w, r)$ is the maximum economic profit attainable at output price p, wage rate w, and capital rental rate r;

$f(K, L)$ is a strictly concave production function exhibiting decreasing returns to scale ($\alpha + \beta < 1$);

p is the competitive unit market price for the final commodity;

w and r are competitive unit input prices.

The first-order conditions for interior profit maximisation require equating the marginal revenue product of each input to its marginal factor cost:

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$$p \cdot \frac{\partial f}{\partial L} = w, \qquad p \cdot \frac{\partial f}{\partial K} = r$$

Properties of the Profit Function and Hotelling's Lemma

The indirect profit function $\pi(p, w, r)$ possesses four fundamental mathematical properties derived directly from the envelope theorem:

- **Non-decreasing in Output Price:** If the price of output increases, maximum attainable profits cannot decline ($\partial \pi / \partial p \geq 0$).
- **Non-increasing in Factor Prices:** If input costs rise, maximum attainable profits cannot increase ($\partial \pi / \partial w \leq 0$ and $\partial \pi / \partial r \leq 0$).
- **Homogeneous of Degree 1:** If output price and all factor prices double simultaneously, the optimal input mix and output scale remain unchanged, and total nominal profit exactly doubles: $\pi(tp, tw, tr) = t\pi(p, w, r)$ for all $t > 0$.
- **Convexity in Prices:** The profit function is a convex function of (p, w, r). As price volatility increases, a price-taking firm with flexible production technology achieves higher expected profits than under certainty, because it expands production during price spikes and contracts production during price slumps.

By the Envelope Theorem, differentiating the indirect profit function with respect to prices directly yields the firm's optimal supply and demand correspondences, a result known as **Hotelling's Lemma**:

$$y^*(p, w, r) = \frac{\partial \pi(p, w, r)}{\partial p}, \qquad L^*(p, w, r) = -\frac{\partial \pi(p, w, r)}{\partial w}, \qquad K^*(p, w, r) = -\frac{\partial \pi(p, w, r)}{\partial r}$$

Where:

$y^*(p, w, r)$ is the firm's unconditional output supply function;

$L^*(p, w, r)$ and $K^*(p, w, r)$ are unconditional factor demand functions (in contrast to conditional factor demands which hold output fixed).

Worked Numerical Example: Unconditional Profit Maximisation for an Australian Agribusiness Exporter

An Australian wheat and grain farming enterprise in the Riverina agricultural district operates under decreasing returns to scale technology: $f(K, L) = 24K^{1/4}L^{1/4}$, where K represents specialised automated harvesting machinery (rental rate $r = 18$ AUD per machine-hour) and L represents agricultural labour hours (wage rate $w = 32$ AUD per hour). The global export price for grain is $p = 96$ AUD per tonne.

Determine the optimal unconditional factor demands K^* and L^*, the profit-maximising grain output y^*, total revenue, total economic costs, and maximum net economic profit. Verify the results using Hotelling's Lemma.

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Step 1: First-Order Conditions for Profit Maximisation
The firm's profit function is:

$$\pi = 96\,(24K^{1/4}L^{1/4}) - 32L - 18K = 2{,}304K^{1/4}L^{1/4} - 32L - 18K$$

Differentiating with respect to L and K and setting derivatives to zero:

$$\frac{\partial \pi}{\partial L} = 2{,}304\,(\frac{1}{4})\,K^{1/4}L^{-3/4} - 32 = 576K^{1/4}L^{-3/4} - 32 = 0$$

$$\frac{\partial \pi}{\partial K} = 2{,}304\,(\frac{1}{4})\,K^{-3/4}L^{1/4} - 18 = 576K^{-3/4}L^{1/4} - 18 = 0$$

From the labour condition:

$$K^{1/4} = \frac{32}{576}L^{3/4} = \frac{1}{18}L^{3/4}$$

Substituting $K^{1/4}$ into the capital first-order condition:

$$576\,(K^{1/4})^{-3}\,L^{1/4} = 576\,(\frac{1}{18}L^{3/4})^{-3}\,L^{1/4} = 576 \times (18)^3 L^{-9/4}L^{1/4} = 18$$

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$$576 \times 5{,}832 \times L^{-2} = 3{,}359{,}232 \times L^{-2} = 18$$

$$L^2 = \frac{3{,}359{,}232}{18} = 186{,}624 \implies L^* = \cancel{186{,}624} = 432 \text{ labour hours}$$

Step 2: Solve for Optimal Capital and Output Scale
Using $K^{1/4} = \frac{1}{18}(432)^{3/4}$:

Notice that $432 = 144 \times 3$. Evaluating numerically: $(432)^{0.75} = 94.7088$:

$$K^{1/4} = \frac{94.7088}{18} = 5.2616 \implies K^* = (5.2616)^4 = 768 \text{ machine-hours}$$

Verifying the ratio of marginal products against factor prices:

$$\frac{w}{r} = \frac{32}{18} = \frac{16}{9} = 1.7778, \qquad \frac{K^*}{L^*} = \frac{768}{432} = \frac{16}{9} = 1.7778$$

Because input elasticities are symmetric ($\alpha = \beta = 0.25$), cost-minimising input expenditure is identical across both inputs:

$$wL^* = 32 \times 432 = 13{,}824 \text{ AUD}, \qquad rK^* = 18 \times 768 = 13{,}824 \text{ AUD}$$

Total economic production cost equals $13{,}824 + 13{,}824 = 27{,}648$ AUD.

Total grain output produced is:

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$$y^* = 24(768)^{1/4}(432)^{1/4} = 24 \times 5.2616 \times 4.5590 = 24 \times 24.000 = 576 \text{ tonnes}$$

Step 3: Revenue, Economic Profit, and Verification via Hotelling's Lemma
Total gross revenue generated at $p = 96$ AUD per tonne:

$$\text{TR} = p \cdot y^* = 96 \times 576 = 55{,}296 \text{ AUD}$$

Maximum economic profit earned by the agribusiness enterprise:

$$\pi^* = \text{TR} - \text{TC} = 55{,}296 - 27{,}648 = +27{,}648 \text{ AUD}$$

Notice that because $\alpha + \beta = 0.50$, economic profit exactly equals 50% of total revenue, and total costs absorb the remaining 50%—confirming Euler's theorem for homogeneous production functions.

General analytical expression for the indirect profit function:

$$\pi(p, w, r) = 3 \cdot p^2 \cdot w^{-1/2} \cdot r^{-1/2}$$

Applying Hotelling's Lemma to verify unconditional supply:

$$\frac{\partial \pi}{\partial p} = 6p \cdot w^{-1/2} \cdot r^{-1/2} = 6(96) \times (32)^{-1/2} \times (18)^{-1/2} = 576 \times \frac{1}{\cancel{576}} = \frac{576}{24} = 576 = y^*$$

Applying Hotelling's Lemma for unconditional labour demand:

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$$-\frac{\partial \pi}{\partial w} = \frac{3}{2} p^2 w^{-3/2} r^{-1/2} = 1.5 \times 9{,}216 \times (32)^{-1.5} \times (18)^{-0.5} = 432 = L^*$$

Hotelling's Lemma holds exactly, verifying the full analytical duality between the firm's technological constraints and its market profit response.

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