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Bachelor of Commerce (B2001) | Level 2 • Part B Major | Monash University

Intermediate microeconomics

Course Code: ECC2000 • Module 1 Notes

Module 1: Preferences, Utility Maximisation, Expenditure Minimisation & Slutsky Decomposition

Chapter 1: Rational Preferences, Axioms of Choice, and Revealed Preference

1.1 Choice Sets and Binary Preference Relations

Consumer theory at the intermediate level begins not with utility numbers or prices, but with an observable choice environment and a primitive binary relation representing how an individual ranks alternatives. Let the consumption set be $X = \mathrm{R}^n_+$, representing non-negative bundles of n distinct commodities. For a two-good economy, a consumption bundle is a vector $x = (x_1, x_2) \in \mathrm{R}^2_+$, where $x_1 \geq 0$ and $x_2 \geq 0$ measure physical quantities consumed per period—for instance, weekly tram journeys across Melbourne and litres of specialty coffee, or hours of leisure and composite household expenditure measured in Australian dollars.

Given any two bundles $x, y \in X$, the consumer's preferences are summarised by the **weak preference relation** $\succsim$, where $x \succsim y$ is read as "bundle x is at least as good as bundle y." From this single primitive relation, two derived relations follow immediately:

- **Strict preference ($\succ$):** $x \succ y$ holds if and only if $x \succsim y$ and it is not the case that $y \succsim x$. The consumer strictly prefers x to y.
- **Indifference ($\sim$):** $x \sim y$ holds if and only if $x \succsim y$ and $y \succsim x$. The consumer views x and y as equally desirable.

Notice that preferences are purely ordinal comparisons between bundles. At this stage, no assumption is made about psychological pleasure, cardinal intensity of satisfaction, or interpersonal comparisons of well-being.

1.2 Axioms of Rational Choice and Regularity Conditions

To construct a mathematically tractable model of consumer demand, microeconomic theory imposes core rationality axioms and technical regularity assumptions on $\succsim$. The first three define **rationality**, while the remaining conditions ensure smooth indifference curves and interior optima.

Axiom / Assumption	Formal Statement (for all $x, y, z \in X$)	Economic Interpretation & Role
Completeness	Either $x \succsim y$, $y \succsim x$, or both.	The consumer can rank every pair of bundles without indecisiveness.
Transitivity	If $x \succsim y$ and $y \succsim z$, then $x \succsim z$.	Prevents preference cycles ($x \succ y \succ z \succ x$) that would make an agent vulnerable to a "money pump."

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Axiom / Assumption	Formal Statement (for all $x, y, z \in X$)	Economic Interpretation & Role
Reflexivity	$x \gtrsim x$ for every $x \in X$.	Any bundle is at least as good as itself (implied automatically by completeness).
Continuity	Upper and lower contour sets $\{y : y \gtrsim x\}$ and $\{y : x \gtrsim y\}$ are closed in X.	Rules out sudden lexicographic jumps; guarantees a continuous utility representation exists.
Local Non-Satiation (LNS)	For any x and any $\epsilon > 0$, there exists y with $\|y - x\| < \epsilon$ such that $y \succ x$.	Slightly more of at least one desirable good improves welfare; guarantees the budget binds at optimum (Walras' Law).
Strong Monotonicity	If $y \geq x$ and $y \neq x$, then $y \succ x$.	"More is strictly better": indifference curves must slope downward and cannot be thick bands.
Strict Convexity	If $x \sim y$ and $x \neq y$, then $tx + (1-t)y \succ x$ for all $t \in (0,1)$.	Averages are preferred to extremes; guarantees unique demand bundles and diminishing MRS.

Local non-satiation is strictly weaker than monotonicity: it permits goods to be "bads" over some ranges provided there is always a direction of slight improvement nearby. Strict convexity captures the empirical observation that consumers diversify consumption rather than spending their entire income on a single commodity, and mathematically ensures that first-order tangency conditions locate a unique global maximum rather than multiple or boundary optima.

1.3 Indifference Curves and the Marginal Rate of Substitution

Given a reference bundle $\bar{x} \in \mathrm{R}^2_+$, the **indifference set** (or indifference curve in two dimensions) is the locus of all bundles ranked as equally desirable to $\bar{x}$:

$$I(\bar{x}) = \{x \in \mathrm{R}^2_+ : x \sim \bar{x}\}$$

Where:

- $I(\bar{x})$ denotes the indifference curve passing through the reference consumption bundle $\bar{x}$.
- $x = (x_1, x_2)$ denotes any non-negative bundle in the two-good commodity space.

Under completeness and transitivity, two indifference curves corresponding to distinct levels of preference can never intersect. Suppose two curves crossed at bundle B, with $A \sim B$ on one curve and $C \sim B$ on the other, while A contains strictly more of both goods than C. By transitivity, $A \sim C$, contradicting strong monotonicity ($A \succ C$). Under strong monotonicity, indifference curves are downward sloping and shift outward toward higher preference levels in the north-east direction.

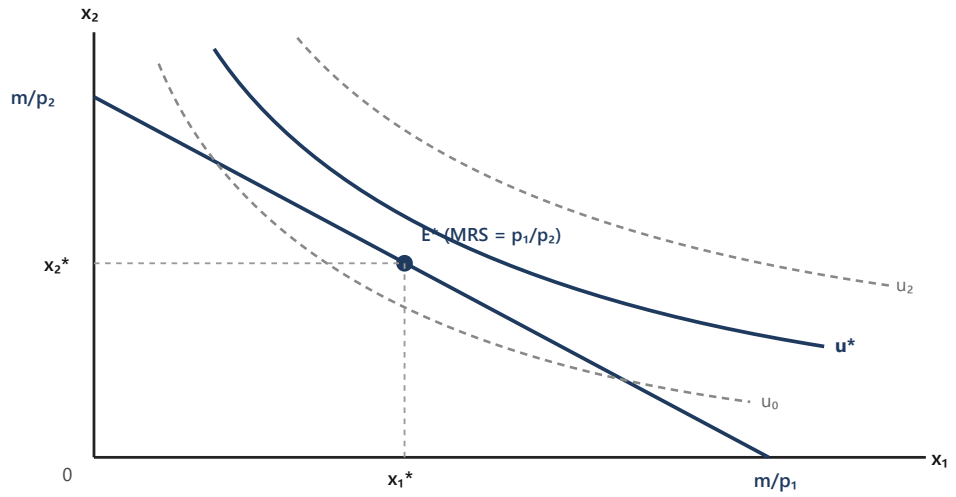
The curvature of an indifference curve is measured by the **Marginal Rate of Substitution ($MRS_{1,2}$)**, defined as the maximum quantity of good 2 a consumer is willing to give up to obtain one additional unit of good 1 while remaining on the same indifference curve. Differentiating along an indifference level $u(x_1, x_2) = \bar{u}$ yields:

$$du = \frac{\partial u}{\partial x_1}dx_1 + \frac{\partial u}{\partial x_2}dx_2 = 0 \implies MRS_{1,2} = -\frac{dx_2}{dx_1}\bigg|_{u=\bar{u}} = \frac{MU_1(x_1,x_2)}{MU_2(x_1,x_2)}$$

Where:

- $MU_1 = \partial u/\partial x_1$ and $MU_2 = \partial u/\partial x_2$ denote the marginal utilities of good 1 and good 2, respectively.
- dx_2/dx_1 is the negative slope of the indifference curve at bundle (x_1, x_2).
- $MRS_{1,2}$ is reported as a positive magnitude representing units of good 2 sacrificed per marginal unit of good 1 gained.

Strict convexity implies **diminishing marginal rate of substitution**: as x_1 increases and x_2 falls along a given indifference curve, $MRS_{1,2}$ strictly decreases. Intuitively, when good 1 is scarce and good 2 is abundant, the consumer is willing to sacrifice many units of good 2 for a single unit of good 1; once good 1 becomes plentiful, willingness to trade good 2 for additional good 1 declines.



1.4 Revealed Preference Theory: WARP and SARP

Because preferences are unobservable psychological primitives, Paul Samuelson developed **Revealed Preference Theory** to test whether observed market choices across varying prices and budgets are consistent with rational utility maximisation. Suppose a consumer chooses bundle x^t at price vector p^t and chooses x^s at price vector p^s ($x^t \; R \; x^s$):

- Bundle x^t is **directly revealed preferred** to x^s ($x^t \gtrsim^D x^s$) if $p^t \cdot x^t \geq p^t \cdot x^s$—meaning x^s was affordable when x^t was chosen. If $p^t \cdot x^t > p^t \cdot x^s$, x^t is strictly directly revealed preferred ($x^t \succ^D x^s$).
- **Weak Axiom of Revealed Preference (WARP):** If x^t is directly revealed preferred to x^s ($p^t \cdot x^t \geq p^t \cdot x^s$) and $x^t \; R \; x^s$, then x^s cannot be strictly directly revealed preferred to x^t; that is, $p^s \cdot x^s < p^s \cdot x^t$ must hold (equivalently, $p^s \cdot x^t > p^s \cdot x^s$ when x^s was chosen).

- **Strong Axiom of Revealed Preference (SARP):** If x^t is indirectly revealed preferred to x^s via a finite chain $x^t \succsim^D x^a \succsim^D \cdots \succsim^D x^s$, then x^s cannot be strictly revealed preferred to x^t.

By Afriat's Theorem, satisfaction of SARP (and its generalised form GARP allowing flat indifference segments) is both necessary and sufficient for observed Australian household scanner data (such as ABS Household Expenditure Survey panels) to be rationalisable by a well-behaved, locally non-satiated, concave utility function.

Chapter 2: Utility Representation and Canonical Functional Forms

2.1 Ordinal Utility and Monotonic Transformations

By Debreu's Representation Theorem, if a preference relation $\succsim$ on $X = \mathrm{R}^n_+$ is complete, transitive, and continuous, then there exists a continuous real-valued **utility function** $u : X \to \mathrm{R}$ that represents $\succsim$ in the sense that:

$$x \succsim y \iff u(x) \geq u(y) \quad \text{for all } x, y \in X$$

Where:

- $u(x)$ is the ordinal utility index assigned to bundle x.
- $x \succsim y$ denotes weak preference of bundle x over bundle y.

Because utility is strictly ordinal, the numerical magnitude of $u(x)$ carries no meaning beyond ranking. Suppose $f : \mathrm{R} \to \mathrm{R}$ is any strictly increasing function ($f'(u) > 0$). Then the composite function $v(x) = f(u(x))$ represents the exact same preference ordering, because $u(x) \geq u(y) \iff f(u(x)) \geq f(u(y))$. Crucially, the MRS is invariant under any strictly increasing differentiable transformation:

$$MRS^{v}_{1,2} = \frac{\partial v / \partial x_1}{\partial v / \partial x_2} = \frac{f'(u) \cdot MU_1}{f'(u) \cdot MU_2} = \frac{MU_1}{MU_2} = MRS^{u}_{1,2}$$

Where:

- $f'(u) > 0$ is the derivative of the monotonic transformation f with respect to u.
- $MRS^{v}_{1,2}$ and $MRS^{u}_{1,2}$ are the marginal rates of substitution under the transformed and original utility functions, respectively.

This property is enormously convenient in applied optimisation. For example, taking the natural logarithm of a Cobb-Douglas utility function $u(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$ transforms a multiplicative objective into an additive, concave objective $v(x_1, x_2) = \alpha \ln x_1 + \beta \ln x_2$ without altering any consumer demand curve.

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2.2 Canonical Utility Functions in Intermediate Microeconomics

Five canonical functional forms dominate theoretical and empirical microeconomics in Australia and internationally:

Preference Class	Utility Function $u(x_1, x_2)$	$MRS_{1,2}$	Elasticity of Substitution (σ)	Key Behavioural Property
Cobb-Douglas	$x_1^{\alpha} x_2^{\beta}$ $(\alpha, \beta > 0)$	$\frac{\alpha x_2}{\beta x_1}$	$\sigma = 1$	Constant budget shares $\frac{\alpha}{\alpha+\beta}$ and $\frac{\beta}{\alpha+\beta}$; homothetic.
Perfect Substitutes	$ax_1 + bx_2$ $(a, b > 0)$	$\frac{a}{b}$ (constant)	$\sigma = \infty$	Linear indifference curves; bang-bang corner solutions unless $p_1/p_2 = a/b$.
Perfect Complements	$\min\{ax_1, bx_2\}$ $(a, b > 0)$	Indeterminate at kink	$\sigma = 0$	L-shaped indifference curves; fixed consumption ratio $ax_1 = bx_2$.
Quasi-Linear	$v(x_1) + x_2$ $(v' > 0, v'' < 0)$	$v'(x_1)$	Varies with x_1	Zero income effect on x_1 at interior optima; $CV = \Delta CS = EV$.
CES	$(ax_1^{\rho} + bx_2^{\rho})^{1/\rho}$ $(\rho \leq 1, \rho \neq 0)$	$\frac{a}{b}\left(\frac{x_2}{x_1}\right)^{1-\rho}$	$\sigma = \frac{1}{1-\rho}$	Nests perfect substitutes ($\rho \to 1$), Cobb-Douglas ($\rho \to 0$), and Leontief ($\rho \to -\infty$).

The **elasticity of substitution (σ)** measures the percentage change in the optimal consumption ratio x_2/x_1 divided by the percentage change in $MRS_{1,2}$ along a given indifference curve:

$$\sigma = \frac{d \ln(x_2/x_1)}{d \ln(MRS_{1,2})}$$

Where:

- x_2/x_1 is the consumption ratio along a fixed indifference level.
- $MRS_{1,2}$ is the marginal rate of substitution evaluated at (x_1, x_2).

Worked Example 2.1: Monotonic Invariance and CES Elasticity of Substitution

Suppose an Australian household has CES preferences over renewable electricity (x_1, in kWh) and natural gas (x_2, in MJ) represented by $u(x_1, x_2) = (x_1^{0.5} + x_2^{0.5})^2$. Here $\rho = 0.5$ and the outer exponent $2 = 1/\rho$.

1. **Step 1 — Monotonic transformation:** Define $v(x_1, x_2) = u(x_1, x_2)^{0.5} = x_1^{0.5} + x_2^{0.5}$. Since $f(u) = u^{0.5}$ has $f'(u) = 0.5u^{-0.5} > 0$ for $u > 0$, v represents identical preferences.

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2. **Step 2 — Compute $MRS_{1,2}$:** Differentiating v gives $MU_1 = 0.5x_1^{-0.5}$ and $MU_2 = 0.5x_2^{-0.5}$. Hence:

$$MRS_{1,2} = \frac{0.5x_1^{-0.5}}{0.5x_2^{-0.5}} = \left(\frac{x_2}{x_1}\right)^{0.5}$$

   Where $MRS_{1,2}$ is the number of MJ of gas the household will give up for one extra kWh of renewable electricity.

3. **Step 3 — Compute σ:** Taking natural logs gives $\ln(MRS_{1,2}) = 0.5\ln(x_2/x_1)$, so $\ln(x_2/x_1) = 2\ln(MRS_{1,2})$. Differentiating yields:

$$\sigma = \frac{d\ln(x_2/x_1)}{d\ln(MRS_{1,2})} = 2 = \frac{1}{1-0.5}$$

   Where $\sigma = 2$ indicates relatively strong substitutability between electricity and gas: a 10% rise in the relative price of gas induces a 20% increase in the electricity-to-gas consumption ratio.

Chapter 3: Constrained Utility Maximisation, Marshallian Demand & Indirect Utility

3.1 The Budget Set and Non-Linear Pricing Constraints

Given strictly positive market prices $p = (p_1, p_2) \gg 0$ and disposable monetary income $m > 0$, the consumer's Walrasian **budget set** is $B(p,m) = \{(x_1,x_2) \in \mathrm{R}^2_+ : p_1x_1 + p_2x_2 \leq m\}$. Under standard linear pricing, the budget line has vertical intercept m/p_2, horizontal intercept m/p_1, and slope $-p_1/p_2$, which measures the market opportunity cost of good 1 in terms of forgone units of good 2.

In Australian retail policy and public economics, consumers frequently encounter **piecewise-linear budget constraints** with kinks. Examples include progressive block tariffs for residential water in Melbourne (where marginal charges jump once quarterly consumption exceeds a threshold $\bar{x}_1$), frequent-flyer bulk discounts, or means-tested income support tapers under Centrelink rules where effective marginal tax rates change discretely at income thresholds. Kinked budget constraints create regions where multiple preference profiles bunch at the kink point.

3.2 The Primal Utility Maximisation Problem and Kuhn-Tucker Conditions

A rational consumer solves the primal Utility Maximisation Problem (UMP):

$$\max_{x_1,x_2 \geq 0} u(x_1,x_2) \quad \text{subject to} \quad p_1x_1 + p_2x_2 \leq m$$

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Where:

- $u(x_1, x_2)$ is a continuously differentiable, strictly quasiconcave utility function satisfying local non-satiation.
- $p_1, p_2 > 0$ are unit prices of goods 1 and 2, and $m > 0$ is nominal income.

By local non-satiation, the budget constraint binds with equality ($p_1 x_1 + p_2 x_2 = m$). Forming the Lagrangian with multiplier $\lambda \geq 0$ (the marginal utility of income):

$$\mathrm{L}(x_1, x_2, \lambda) = u(x_1, x_2) + \lambda(m - p_1 x_1 - p_2 x_2)$$

Where λ represents the shadow price of the budget constraint—the rate at which maximum attainable utility increases per additional dollar of income ($\partial V / \partial m$). Incorporating the non-negativity bounds $x_1 \geq 0, x_2 \geq 0$ via Kuhn-Tucker complementary slackness yields:

$$\frac{\partial u}{\partial x_i} - \lambda p_i \leq 0, \qquad x_i \geq 0, \qquad x_i \left(\frac{\partial u}{\partial x_i} - \lambda p_i\right) = 0 \quad (i = 1, 2)$$

Where:

- $x_i > 0$ implies equality $\partial u / \partial x_i = \lambda p_i$.
- $\partial u / \partial x_i < \lambda p_i$ forces the corner solution $x_i^* = 0$.

At an **interior solution** ($x_1^* > 0, x_2^* > 0$), both first-order conditions hold with equality, giving the classic equi-marginal principle and tangency condition:

$$\frac{MU_1(x_1^*, x_2^*)}{p_1} = \frac{MU_2(x_1^*, x_2^*)}{p_2} = \lambda \iff MRS_{1,2}(x_1^*, x_2^*) = \frac{p_1}{p_2}$$

Where:

- MU_i / p_i is the marginal utility per dollar spent on good i.
- $MRS_{1,2}$ is the subjective marginal willingness to substitute good 2 for good 1, equated to the market price ratio p_1 / p_2.

If $MRS_{1,2}(0, m/p_2) \leq p_1 / p_2$, even the very first marginal unit of good 1 delivers less utility per dollar than good 2; the consumer therefore chooses the boundary bundle $(0, m/p_2)$.

3.3 Marshallian Demand, Indirect Utility, and Roy's Identity

Solving the first-order conditions together with the budget constraint yields the **Marshallian (Walrasian) demand functions** $x_1(p_1, p_2, m)$ and $x_2(p_1, p_2, m)$. Every Marshallian demand system derived from rational utility maximisation satisfies two fundamental properties:

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1. **Homogeneity of degree zero in (p, m):** Scaling all prices and nominal income by any scalar $t > 0$ leaves the budget set unchanged, so $x_i(tp_1, tp_2, tm) = x_i(p_1, p_2, m)$. Rational consumers exhibit no money illusion.
2. **Walras' Law (Budget Exhaustion):** Under local non-satiation, $p_1 x_1(p, m) + p_2 x_2(p, m) \equiv m$ for all $(p, m) \gg 0$.

Substituting the optimal Marshallian demands back into the direct utility function defines the **indirect utility function** $V(p_1, p_2, m)$:

$$V(p_1, p_2, m) = u(x_1(p_1, p_2, m),\, x_2(p_1, p_2, m))$$

Where $V(p_1, p_2, m)$ reports the highest level of utility achievable at prices (p_1, p_2) and income m. The indirect utility function is homogeneous of degree zero in (p, m), strictly increasing in m, non-increasing in p_i, and quasiconvex in p. Applying the Envelope Theorem to the Lagrangian yields **Roy's Identity**, which recovers Marshallian demand directly by differentiating V:

$$x_i(p_1, p_2, m) = -\frac{\partial V(p_1, p_2, m)/\partial p_i}{\partial V(p_1, p_2, m)/\partial m} \quad (i = 1, 2)$$

Where:

- $\partial V / \partial p_i = -\lambda x_i \leq 0$ is the marginal utility loss from a small rise in price p_i.
- $\partial V / \partial m = \lambda > 0$ is the marginal utility of an extra dollar of income.

Worked Example 3.1: Cobb-Douglas Lagrangian, Indirect Utility, and Roy's Identity

Consider a Monash University student allocating a weekly discretionary budget m = \$120 between specialty coffees (x_1, price p_1) and streaming/leisure services (x_2, price p_2), with utility $u(x_1, x_2) = x_1^{1/3} x_2^{2/3}$.

1. **Step 1 — Monotonic log transformation and Lagrangian:** Maximising $v(x_1, x_2) = \frac{1}{3} \ln x_1 + \frac{2}{3} \ln x_2$ subject to $p_1 x_1 + p_2 x_2 = m$ gives:

   $$\mathrm{L} = \frac{1}{3} \ln x_1 + \frac{2}{3} \ln x_2 + \lambda(m - p_1 x_1 - p_2 x_2)$$

   Where λ is the Lagrange multiplier on the weekly budget constraint.
2. **Step 2 — First-order conditions:**

   $$\frac{\partial \mathrm{L}}{\partial x_1} = \frac{1}{3x_1} - \lambda p_1 = 0, \qquad \frac{\partial \mathrm{L}}{\partial x_2} = \frac{2}{3x_2} - \lambda p_2 = 0$$

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Rearranging gives $p_1 x_1 = \frac{1}{3\lambda}$ and $p_2 x_2 = \frac{2}{3\lambda}$.

3. **Step 3 — Solve for λ and Marshallian demands:** Summing expenditure gives $p_1 x_1 + p_2 x_2 = \frac{1}{\lambda} = m$, so $\lambda = 1/m$. Substituting back yields:

$$x_1(p_1, p_2, m) = \frac{m}{3p_1}, \qquad x_2(p_1, p_2, m) = \frac{2m}{3p_2}$$

At m = \$120, p_1 = \$5, and p_2 = \$10, the student buys $x_1^* = 120/(3 \times 5) = 8$ coffees and $x_2^* = 240/(3 \times 10) = 8$ leisure units.

4. **Step 4 — Indirect utility and Roy's Identity check:** Substituting into the original $u(x_1, x_2)$ gives:

$$V(p_1, p_2, m) = \left(\frac{m}{3p_1}\right)^{1/3} \left(\frac{2m}{3p_2}\right)^{2/3} = \frac{2^{2/3}}{3} m p_1^{-1/3} p_2^{-2/3}$$

Differentiating V with respect to p_1 and m:

$$\frac{\partial V}{\partial p_1} = -\frac{1}{3} \cdot \frac{2^{2/3}}{3} m p_1^{-4/3} p_2^{-2/3}, \qquad \frac{\partial V}{\partial m} = \frac{2^{2/3}}{3} p_1^{-1/3} p_2^{-2/3}$$

Dividing $-(\partial V / \partial p_1)$ by $(\partial V / \partial m)$ recovers $x_1 = \frac{m}{3p_1}$, confirming Roy's Identity.

Worked Example 3.2: Quasi-Linear Utility Interior vs Corner Solution

Suppose a Melbourne commuter has quasi-linear preferences over express train upgrades (x_1, price p_1) and composite spending in dollars (x_2, numeraire $p_2 = 1$), given by $u(x_1, x_2) = 12\sqrt{x_1} + x_2$ with weekly income m.

1. **Step 1 — Interior tangency condition:** Setting $MRS_{1,2} = MU_1/MU_2 = 6x_1^{-1/2} = p_1/1$ gives $x_1^{1/2} = 6/p_1$, or:

$$x_1^*(p_1, m) = \frac{36}{p_1^2}$$

Notice x_1^* is completely independent of income m whenever $x_2^* > 0$.

2. **Step 2 — Check the non-negativity boundary for x_2:** Expenditure on good 1 at the interior formula is $p_1 x_1^* = 36/p_1$. Thus $x_2^* = m - 36/p_1$, which requires $m \geq 36/p_1$.

3. **Step 3 — Numerical evaluation:** If p_1 = \$3 and m = \$50, then $x_1^* = 36/9 = 4$ upgrades and $x_2^* = 50 - 12 = 38$ dollars. However, if m = \$8 and p_1 = \$3, the interior formula demands

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spending \$12 > \$8, which is infeasible. The consumer hits the corner $x_2^* = 0$ and spends all income on good 1: $x_1^* = 8/3 \approx 2.67$.

Chapter 4: Expenditure Minimisation, Hicksian Demand, Duality & Slutsky Decomposition

4.1 The Dual Expenditure Minimisation Problem and Shephard's Lemma

Whereas the primal problem maximises utility for a given budget, the dual **Expenditure Minimisation Problem (EMP)** finds the cheapest bundle that achieves a target utility level $\bar{u}$ at prices $p = (p_1, p_2) \gg 0$:

$$e(p_1, p_2, \bar{u}) = \min_{x_1, x_2 \geq 0} p_1 x_1 + p_2 x_2 \quad \text{subject to} \quad u(x_1, x_2) \geq \bar{u}$$

Where:

- $e(p_1, p_2, \bar{u})$ is the **expenditure function**, giving the minimum dollar cost of attaining utility $\bar{u}$.
- $h_1(p_1, p_2, \bar{u})$ and $h_2(p_1, p_2, \bar{u})$ are the **Hicksian (compensated) demand functions** solving the EMP.

Forming the dual Lagrangian $\mathrm{L}^d = p_1 x_1 + p_2 x_2 + \mu(\bar{u} - u(x_1, x_2))$ yields the identical interior tangency condition $MRS_{1,2} = p_1/p_2$, now paired with the utility constraint $u(x_1, x_2) = \bar{u}$. The expenditure function satisfies four fundamental mathematical properties:

- **Homogeneity of degree 1 in p:** $e(tp, \bar{u}) = te(p, \bar{u})$ for $t > 0$ (doubling prices doubles minimum expenditure).
- **Strictly increasing in $\bar{u}$ and non-decreasing in p:** Higher utility requires more spending; higher prices cannot lower minimum expenditure.
- **Concavity in p:** Because a minimum of linear functions $p \cdot x$ over a fixed upper contour set is concave, $e(p, \bar{u})$ is concave in prices.
- **Shephard's Lemma:** Wherever $e(p, \bar{u})$ is differentiable with respect to p_i, its partial derivative equals the Hicksian demand:

$$h_i(p_1, p_2, \bar{u}) = \frac{\partial e(p_1, p_2, \bar{u})}{\partial p_i} \quad (i = 1, 2)$$

  Where $h_i(p, \bar{u})$ is the compensated demand for good i at target utility $\bar{u}$.

By concavity of $e(p, \bar{u})$ in p, its second derivative with respect to p_i is non-positive: $\frac{\partial h_i}{\partial p_i} = \frac{\partial^2 e}{\partial p_i^2} \leq 0$. This establishes the **Compensated Law of Demand**: Hicksian own-price demand curves can never slope upward. Furthermore, symmetry of cross-partial derivatives (Young's Theorem) implies symmetry of cross-substitution effects: $\frac{\partial h_i}{\partial p_j} = \frac{\partial^2 e}{\partial p_i \partial p_j} = \frac{\partial h_j}{\partial p_i}$.

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4.2 Duality Identities and the Slutsky Equation

When u is continuous, locally non-satiated, and strictly quasiconcave, the primal and dual problems are connected by four exact duality identities:

1. $e(p, V(p,m)) \equiv m$ (minimising expenditure to reach the maximum utility affordable at m costs exactly m).
2. $V(p, e(p,\bar{u})) \equiv \bar{u}$ (the highest utility affordable when spending $e(p,\bar{u})$ is exactly $\bar{u}$).
3. $x_i(p,m) \equiv h_i(p, V(p,m))$ (Marshallian demand equals Hicksian demand evaluated at attained utility).
4. $h_i(p,\bar{u}) \equiv x_i(p, e(p,\bar{u}))$ (Hicksian demand equals Marshallian demand when income is adjusted to $e(p,\bar{u})$).

Differentiating identity (4) with respect to p_j and evaluating at $\bar{u} = V(p,m)$ (so $e(p,\bar{u}) = m$ and $h_j = x_j$ via Shephard's Lemma) yields the **Slutsky Equation**:

$$\frac{\partial x_i(p,m)}{\partial p_j} = \frac{\partial h_i(p,\bar{u})}{\partial p_j} - x_j(p,m)\frac{\partial x_i(p,m)}{\partial m}$$

Where:

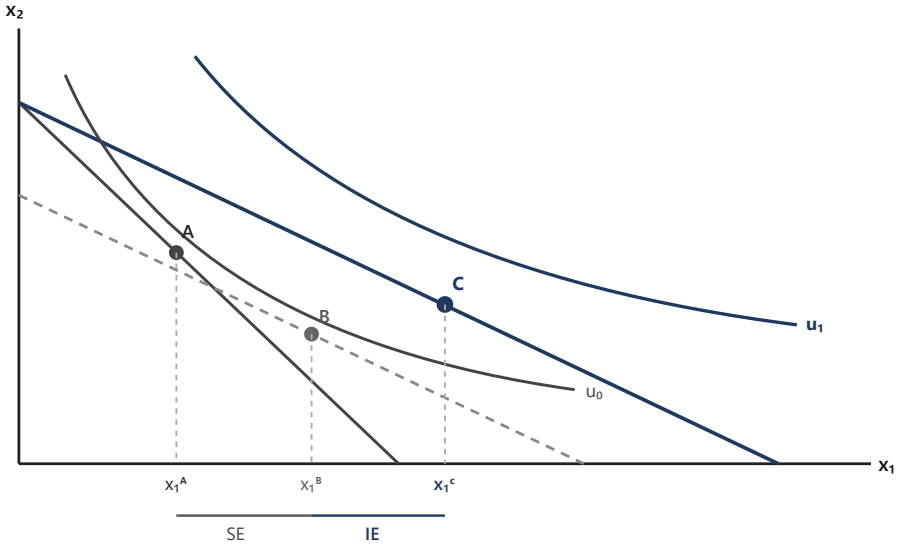
- $\frac{\partial x_i}{\partial p_j}$ is the **total price effect** on Marshallian demand for good i when price p_j changes.
- $\frac{\partial h_i}{\partial p_j}$ is the **substitution effect** (always ≤ 0 when $i = j$), holding utility constant at $\bar{u}$.
- $-x_j \frac{\partial x_i}{\partial m}$ is the **income effect**, capturing the loss in real purchasing power ($dm = -x_j dp_j$) multiplied by the income responsiveness $\partial x_i / \partial m$.

When $i = j$, the own-price Slutsky equation $\frac{\partial x_1}{\partial p_1} = \frac{\partial h_1}{\partial p_1} - x_1 \frac{\partial x_1}{\partial m}$ explains the complete taxonomy of goods:

- **Normal good ($\partial x_1/\partial m > 0$):** Both substitution ($\leq 0$) and income ($-x_1 \partial x_1/\partial m < 0$) effects push demand downward when p_1 rises, guaranteeing a downward-sloping Marshallian demand curve.
- **Ordinary inferior good ($\partial x_1/\partial m < 0$, small in magnitude):** The income effect is positive ($-x_1 \partial x_1/\partial m > 0$) and partially offsets the negative substitution effect, yet the substitution effect dominates so $\partial x_1/\partial p_1 < 0$.
- **Giffen good ($\partial x_1/\partial m < 0$, large in magnitude):** The good is so strongly inferior and consumes such a large budget share that the positive income effect outweighs the negative substitution effect, yielding $\partial x_1/\partial p_1 > 0$. Every Giffen good must be inferior, though most inferior goods are not Giffen.

For discrete price changes from p_1 to p_1', microeconomics distinguishes two compensation concepts: **Hicksian compensation** shifts the new budget line parallel until it is tangent to the original indifference curve u_0, whereas **Slutsky compensation** pivots the budget line through the original bundle (x_1^0, x_2^0) by setting compensated income $m' = p_1' x_1^0 + p_2 x_2^0 = m + x_1^0 \Delta p_1$. For infinitesimal changes, both decompositions coincide with the Slutsky differential equation.

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4.3 Exact Welfare Measures: Compensating Variation, Equivalent Variation, and Consumer Surplus

When a price change moves prices from p^0 to p^1 while holding income fixed at m, ordinary Marshallian Consumer Surplus ($\Delta CS = \int_{p_1^0}^{p_1^1} x_1(p, m)\, dp_1$) is only an exact money metric of welfare when the marginal utility of income is constant (as in quasi-linear preferences). For general preferences, exact Hicksian welfare evaluation relies on the expenditure function:

- **Compensating Variation (CV):** The dollar amount the consumer would need to receive (or pay) at the *new* prices p^1 to remain at the *original* utility level $u^0 = V(p^0, m)$:

  $$CV = e(p^1, u^0) - e(p^0, u^0) = e(p^1, u^0) - m$$

  Where positive CV (for a price increase) measures the compensation required to restore the consumer to u^0.

- **Equivalent Variation (EV):** The dollar amount the consumer would be willing to pay (or accept) at the *old* prices p^0 to avoid the price change and achieve the *new* utility level $u^1 = V(p^1, m)$:

  $$EV = e(p^0, u^0) - e(p^0, u^1) = m - e(p^0, u^1)$$

  Where EV uses fixed base prices p^0 and therefore ranks multiple policy proposals consistently.

For a normal good whose price rises ($p_1^1 > p_1^0$), the ordering of welfare loss magnitudes is always $EV < \Delta CS < CV$. When preferences are quasi-linear in good 2, the income effect on good 1 is zero, Marshallian and Hicksian demands coincide ($x_1 = h_1$), and all three measures are identical: $EV = \Delta CS = CV$.

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Worked Example 4.1: Expenditure Function, Shephard's Lemma, Slutsky Decomposition & CV/EV

Continuing with the Cobb-Douglas preferences $u(x_1, x_2) = x_1^{1/3} x_2^{2/3}$, suppose initial prices are p_1^0 = \$4, p_2 = \$1, and weekly income is m = \$120.

1. **Step 1 — Derive the expenditure function $e(p_1, p_2, \bar{u})$:** Inverting $V(p_1, p_2, m) = \frac{2^{2/3}}{3} m p_1^{-1/3} p_2^{-2/3} = \bar{u}$ for m yields:

$$e(p_1, p_2, \bar{u}) = \frac{3}{2^{2/3}} \bar{u}\, p_1^{1/3} p_2^{2/3}$$

   Where $\bar{u}$ is the target utility level.

2. **Step 2 — Derive Hicksian demand via Shephard's Lemma:**

$$h_1(p_1, p_2, \bar{u}) = \frac{\partial e}{\partial p_1} = \frac{1}{2^{2/3}} \bar{u}\, p_1^{-2/3} p_2^{2/3}$$

   At initial prices (p_1^0 = \$4, p_2 = \$1, m = \$120), $x_1^0 = 120/(3 \times 4) = 10$, $x_2^0 = 240/3 = 80$, and $u^0 = 10^{1/3} 80^{2/3} = 40$. Checking $h_1(4, 1, 40) = 2^{-2/3}(40)(4^{-2/3}) = 40/4 = 10$, confirming duality.

3. **Step 3 — Discrete Slutsky & Hicksian decomposition for a price rise to p_1^1 = \$9:** At p_1^1 = \$9, total Marshallian demand falls to $x_1^1 = 120/(3 \times 9) = 4.44$ units (so $\Delta x_1 = -5.56$).
   - *Hicksian substitution effect:* $h_1(9, 1, 40) = 2^{-2/3}(40)(9^{-2/3}) = 10 \times (4/9)^{2/3} \approx 5.82$. Thus the Hicksian substitution effect is $5.82 - 10 = -4.18$, and the income effect is $4.44 - 5.82 = -1.38$.
   - *Slutsky purchasing-power substitution effect:* Compensated income to buy the old bundle $(10, 80)$ at new prices is $m' = 9(10) + 1(80) = 170$ dollars. At m' = \$170, $x_1(9, 1, 170) = 170/27 \approx 6.30$. Thus the Slutsky substitution effect is $6.30 - 10 = -3.70$, and the Slutsky income effect is $4.44 - 6.30 = -1.86$.
   - *Calculus verification at (p_1^0 = \$4, m = \$120):* $\frac{\partial x_1}{\partial p_1} = -\frac{m}{3p_1^2} = -2.50$. Meanwhile, $\frac{\partial h_1}{\partial p_1} = -\frac{2}{3} h_1 p_1^{-1} = -\frac{2}{3}(10)(0.25) = -1.67$, and $-x_1 \frac{\partial x_1}{\partial m} = -10 \cdot \frac{1}{3(4)} = -0.83$. Summing gives $-1.67 - 0.83 = -2.50$.

4. **Step 4 — Compensating and Equivalent Variation:**
   - $CV = e(9, 1, 40) - 120 = 120 \times (9/4)^{1/3} - 120 = 120(1.3104) - 120 = 37.24$ dollars.
   - New utility is $u^1 = V(9, 1, 120) = 40 \times (4/9)^{1/3} = 30.53$. Thus $e(4, 1, u^1) = 120 \times (4/9)^{1/3} = 91.58$ dollars, giving $EV = 120 - 91.58 = 28.42$ dollars.
   - Marshallian consumer surplus loss is $\Delta CS = \int_4^9 \frac{40}{p_1} dp_1 = 40 \ln(9/4) = 32.44$ dollars. As predicted by theory for a normal good, EV (\$28.42) < ΔCS (\$32.44) < CV (\$37.24).

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4.4 Demand Elasticities and Aggregation Laws

Expressing the Slutsky equation in elasticity form provides a dimensionless bridge between consumer theory and empirical econometrics. Let $\varepsilon_{i,j} = \frac{\partial x_i}{\partial p_j}\frac{p_j}{x_i}$ denote the uncompensated (Marshallian) cross-price elasticity, $\varepsilon^h_{i,j} = \frac{\partial h_i}{\partial p_j}\frac{p_j}{h_i}$ the compensated (Hicksian) elasticity, $\eta_i = \frac{\partial x_i}{\partial m}\frac{m}{x_i}$ the income elasticity, and $s_j = \frac{p_j x_j}{m}$ the budget share of good j. Multiplying the Slutsky equation by p_j/x_i yields:

$$\varepsilon_{i,j} = \varepsilon^h_{i,j} - s_j \eta_i$$

Where:

- $\varepsilon_{i,j}$ is the percentage change in Marshallian demand for good i per 1% rise in p_j.
- $\varepsilon^h_{i,j}$ is the pure compensated price elasticity along a fixed indifference curve.
- $s_j \eta_i$ is the budget-share-weighted income elasticity adjustment.

Moreover, differentiating Walras' Law $p_1 x_1(p,m) + p_2 x_2(p,m) = m$ with respect to m and p_j produces two fundamental consistency restrictions used in Australian demand systems estimation (such as the Almost Ideal Demand System):

- **Engel Aggregation:** $s_1\eta_1 + s_2\eta_2 = 1$ (budget-share-weighted income elasticities sum to unity; not all goods can be inferior, nor can all goods be luxuries).
- **Cournot Aggregation:** $s_1\varepsilon_{1,j} + s_2\varepsilon_{2,j} + s_j = 0$ (accounting for expenditure shifts across all goods following a price change in good j).

4.5 Application: The Labour-Leisure Model and Endowment Income Effects

In Australian labour economics, individuals choose consumption C (price $p_C = 1$) and leisure hours $\ell \in [0, T]$ given a wage rate w and non-labour income M (such as dividend distributions or government transfer payments). Time not spent in leisure is supplied as labour $L = T - \ell$, yielding the full-income budget constraint:

$$C + w\ell = M + wT$$

Where:

- T is the total available time endowment (e.g., 16 hours per day or 112 waking hours per week).
- w is the hourly after-tax wage rate, acting as the opportunity cost (price) of leisure ℓ.
- $M + wT$ is full income, whose value changes automatically whenever w changes because the worker is a net seller of leisure.

Applying the Slutsky equation when the consumer possesses a time endowment T adds the **endowment income effect** $+T\frac{\partial \ell}{\partial m}$ to the standard Slutsky equation, yielding:

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$$\frac{\partial \ell}{\partial w} = \frac{\partial h_\ell}{\partial w} + (T - \ell)\frac{\partial \ell}{\partial m}$$

Where:

- $\frac{\partial h_\ell}{\partial w} \leq 0$ is the pure substitution effect: a higher wage makes leisure costlier, inducing less leisure and more labour supply.
- $(T - \ell)\frac{\partial \ell}{\partial m} > 0$ (assuming leisure is a normal good and labour supply $L = T - \ell > 0$) is the net endowment income effect: a higher wage raises full wealth on hours already worked, inducing more leisure and fewer hours worked.

Because labour supply satisfies $\frac{\partial L}{\partial w} = -\frac{\partial \ell}{\partial w}$, at moderate wages the negative substitution effect on leisure dominates and the labour supply curve slopes upward. At high wages—when hours worked $T - \ell$ are substantial—the positive income effect on leisure can overpower the substitution effect, producing a **backward-bending labour supply curve**.

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