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Bachelor of Commerce (B2001) | Level 3 • Part B Major | Monash University

Investments

Course Code: BFC3241 • Module 2 Notes

Module 2: Optimal Risky Portfolios, Index Models, CAPM, APT & Multi-Factor Models

2.1 Two-Asset Portfolio Theory and Diversification Mechanics

Modern Portfolio Theory (MPT), pioneered by Harry Markowitz (1952), demonstrates that the risk of an individual security cannot be evaluated in isolation; rather, its contribution to portfolio volatility depends critically on how its returns co-move with all other holdings. Consider a risky portfolio P constructed by investing weight w_1 in Asset 1 and weight $w_2 = 1 - w_1$ in Asset 2. The portfolio's expected return is simply the weighted average of the constituent expected returns:

$$E(r_P) = w_1 E(r_1) + w_2 E(r_2)$$

Where: $E(r_P)$ is the portfolio expected return; w_1 and w_2 are the portfolio weights summing to 1.0; and $E(r_1)$ and $E(r_2)$ are the individual expected returns.

Unlike expected return, portfolio variance is *not* a simple weighted average of individual variances. Instead, portfolio variance depends on the pairwise **covariance** $\text{Cov}(r_1, r_2) = \sigma_{12}$ or, equivalently, the dimensionless **correlation coefficient** $\rho_{12} = \frac{\sigma_{12}}{\sigma_1 \sigma_2} \in [-1, +1]$:

$$\sigma_P^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1 w_2 \text{Cov}(r_1, r_2) = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1 w_2 \rho_{12}\sigma_1\sigma_2$$

Where: σ_P^2 is the two-asset portfolio variance; σ_1^2 and σ_2^2 are the individual asset variances; $\text{Cov}(r_1, r_2)$ is the covariance between Asset 1 and Asset 2; and ρ_{12} is their correlation coefficient.

This variance equation explains the mathematical foundation of **diversification** across three benchmark correlation regimes (assuming positive weights $w_1, w_2 > 0$):

- **Perfect Positive Correlation ($\rho_{12} = +1$):** Portfolio variance factors into $(w_1\sigma_1 + w_2\sigma_2)^2$, so $\sigma_P = w_1\sigma_1 + w_2\sigma_2$. The portfolio opportunity set forms a straight line connecting Asset 1 and Asset 2 in $(E(r), \sigma)$ space, offering zero diversification benefit.
- **Imperfect Correlation ($-1 < \rho_{12} < +1$):** Whenever $\rho_{12} < 1$, $\sigma_P < w_1\sigma_1 + w_2\sigma_2$, bowing the investment opportunity curve leftward (hyperbolic curvature) and reducing portfolio standard deviation below the weighted average of constituent volatilities.
- **Perfect Negative Correlation ($\rho_{12} = -1$):** Portfolio variance becomes $(w_1\sigma_1 - w_2\sigma_2)^2$, allowing an investor to construct a zero-variance riskless hedge by choosing $w_1 = \frac{\sigma_2}{\sigma_1 + \sigma_2}$ and $w_2 = \frac{\sigma_1}{\sigma_1 + \sigma_2}$.

Global Minimum Variance Portfolio (GMVP)

Among all feasible combinations of two risky assets, the portfolio that achieves the lowest possible standard deviation regardless of expected return is the **Global Minimum Variance Portfolio (GMVP)**. Differentiating σ_P^2 with respect to w_1 (after substituting $w_2 = 1 - w_1$) and setting the first-order condition to zero yields the analytical GMVP weights:

$$w_{\text{GMV},1} = \frac{\sigma_2^2 - \text{Cov}(r_1, r_2)}{\sigma_1^2 + \sigma_2^2 - 2\text{Cov}(r_1, r_2)}, \qquad w_{\text{GMV},2} = 1 - w_{\text{GMV},1}$$

Where: $w_{\text{GMV},1}$ and $w_{\text{GMV},2}$ are the weights in Asset 1 and Asset 2 that minimise portfolio variance; σ_1^2 and σ_2^2 are the individual asset variances; and $\text{Cov}(r_1, r_2) = \rho_{12}\sigma_1\sigma_2$.

Any portfolio along the two-asset curve lying below the GMVP is **mean-variance inefficient** because an investor can achieve both higher expected return and identical or lower standard deviation by moving to the upper branch of the hyperbola above the GMVP.

Correlation Regime (ρ_{12})	GMVP Weight in Asset 1 ($w_{\text{GMV},1}$)	Minimum Volatility (σ_{GMV})	Geometric Shape in $(E(r), \sigma)$ Space
$\rho_{12} = +1.00$ (No Diversification)	100% in lower-risk asset (if shorting prohibited)	$\sigma_{\text{GMV}} = \min(\sigma_1, \sigma_2)$	Straight line segment connecting Asset 1 and Asset 2
$\rho_{12} = +0.25$ (Moderate Comovement)	$w_{\text{GMV},1} = \frac{\sigma_2^2 - \rho_{12}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}$	Strictly below $\min(\sigma_1, \sigma_2)$	Smooth hyperbola bowing leftward toward the vertical axis
$\rho_{12} = 0.00$ (Uncorrelated)	$w_{\text{GMV},1} = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$	$\sigma_{\text{GMV}} = \frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2}}$	Pronounced leftward curvature with substantial risk reduction

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Correlation Regime (ρ_{12})	GMVP Weight in Asset 1 ($w_{\text{GMV},1}$)	Minimum Volatility (σ_{GMV})	Geometric Shape in $(E(r), \sigma)$ Space
$\rho_{12} = -1.00$ (Perfect Negative)	$w_{\text{GMV},1} = \frac{\sigma_2}{\sigma_1 + \sigma_2}$	$\sigma_{\text{GMV}} = 0.00\%$ (Riskless Hedge)	Two straight-line segments meeting on the vertical $E(r)$ axis

Worked Example 1: Global Minimum Variance Portfolio on the ASX (CBA & RIO)

An Australian equity analyst models two major S&P/ASX 20 constituents: Commonwealth Bank of Australia (CBA, Asset 1) with expected return $E(r_1) = 9.00\%$ and volatility $\sigma_1 = 16.00\%$, and Rio Tinto Limited (RIO, Asset 2) with expected return $E(r_2) = 13.00\%$ and volatility $\sigma_2 = 24.00\%$. The return correlation between CBA and RIO is $\rho_{12} = 0.25$. Compute the GMVP weights, expected return, and standard deviation.

1. **Compute variances and covariance:**

$$\sigma_1^2 = (0.1600)^2 = 0.0256, \quad \sigma_2^2 = (0.2400)^2 = 0.0576$$

$$\text{Cov}(r_1, r_2) = \rho_{12}\sigma_1\sigma_2 = 0.25 \times 0.1600 \times 0.2400 = 0.0096$$

2. **Solve for the GMVP weights:**

$$w_{\text{GMV},1} = \frac{0.0576 - 0.0096}{0.0256 + 0.0576 - 2(0.0096)} = \frac{0.0480}{0.0640} = 0.7500 \text{ (75.00\% in CBA)}$$

$$w_{\text{GMV},2} = 1 - 0.7500 = 0.2500 \text{ (25.00\% in RIO)}$$

3. **Calculate the GMVP expected return and standard deviation:**

$$E(r_{\text{GMV}}) = 0.75(9.00\%) + 0.25(13.00\%) = 6.75\% + 3.25\% = 10.00\%$$

$$\sigma_{\text{GMV}}^2 = (0.75)^2(0.0256) + (0.25)^2(0.0576) + 2(0.75)(0.25)(0.0096) = 0.0144 + 0.0036 + 0.0036 = 0.0216$$

$$\sigma_{\text{GMV}} = \sqrt{0.0216} = 14.70\%$$

Notice that adding 25% of the higher-volatility mining stock (RIO, 24.00%) to CBA (16.00%) *reduces* overall portfolio volatility to 14.70% while simultaneously increasing expected return from 9.00% to 10.00%.

2.2 Optimal Risky Portfolios, Markowitz Frontier, and Two-Fund Separation

When a risk-free asset yielding r_f is available alongside risky assets, the investor seeks the single risky portfolio P^* on the efficient frontier that generates the steepest possible Capital Allocation Line (CAL)—that is, the risky portfolio that **maximises the Sharpe Ratio**:

$$\max_{w_1, w_2} S_P = \frac{E(r_P) - r_f}{\sigma_P} \quad \text{subject to } w_1 + w_2 = 1$$

Where: S_P is the portfolio Sharpe ratio; $E(r_P) - r_f$ is expected excess return; and σ_P is portfolio standard deviation.

Solving this constrained optimisation for two risky assets produces the closed-form weights for the **optimal risky (tangency) portfolio** P^*, where $E(R_i) = E(r_i) - r_f$ denotes each asset's excess return:

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$$w_1^* = \frac{E(R_1)\sigma_2^2 - E(R_2)\text{Cov}(r_1, r_2)}{E(R_1)\sigma_2^2 + E(R_2)\sigma_1^2 - [E(R_1) + E(R_2)]\,\text{Cov}(r_1, r_2)}, \qquad w_2^* = 1 - w_1^*$$

Where: w_1^* and w_2^* are the tangency portfolio weights; $E(R_1) = E(r_1) - r_f$ and $E(R_2) = E(r_2) - r_f$ are expected excess returns above the risk-free rate; σ_1^2 and σ_2^2 are asset variances; and $\text{Cov}(r_1, r_2)$ is their covariance.

The N-Asset Markowitz Frontier and Tobin's Separation Property

Extending the two-asset framework to N risky securities (such as the 200 constituents of the S&P/ASX 200), portfolio variance is expressed compactly in matrix notation as $\sigma_P^2 = \mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w}$, where $\mathbf{w}$ is the $N \times 1$ vector of weights summing to one ($\mathbf{w}^\top \mathbf{1} = 1$) and $\boldsymbol{\Sigma}$ is the $N \times N$ variance-covariance matrix. In summation form, an N-asset portfolio's variance separates into N individual variance terms and $N(N-1)$ covariance terms:

$$\sigma_P^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j \neq i}^{N} w_i w_j \text{Cov}(r_i, r_j)$$

Where: w_i and w_j are the portfolio weights in securities i and j; σ_i^2 is security i's variance; and $\text{Cov}(r_i, r_j)$ is the pairwise covariance. As N grows large in an equally weighted portfolio ($w_i = 1/N$), the N variance terms contribute $\frac{1}{N}\bar{\sigma}^2 \to 0$, whereas the $N(N-1)$ covariance terms converge to the average pairwise covariance $\overline{\text{Cov}}$—proving that systematic co-movement sets the irreducible floor on diversification.

Tracing the lowest possible variance for every target expected return μ_0 generates the **Minimum-Variance Frontier**. The upper segment lying above the GMVP constitutes the **Markowitz Efficient Frontier**. In institutional practice across Australian superannuation funds, quadratic optimisers impose linear inequality constraints—most notably non-negativity constraints ($w_i \geq 0$ for long-only mandates) and sector/single-issuer concentration caps—to prevent unrealistic short positions and control estimation error.

Worked Example 2A: Three-Asset Multi-Class Portfolio Variance via Covariance Matrix

Before extending to the tangency portfolio, consider how an Australian multi-asset manager computes portfolio expected return and volatility across three broad asset classes: Australian Equities ($w_1 = 0.40, E(r_1) = 9.50\%, \sigma_1 = 16.00\%$), Global Equities ($w_2 = 0.35, E(r_2) = 10.50\%, \sigma_2 = 15.00\%$), and Australian Fixed Income ($w_3 = 0.25, E(r_3) = 4.50\%, \sigma_3 = 5.00\%$). Pairwise correlations are $\rho_{12} = 0.70$, $\rho_{13} = 0.10$, and $\rho_{23} = 0.15$.

Covariance Matrix $\boldsymbol{\Sigma}$	Australian Equities ($w_1 = 0.40$)	Global Equities ($w_2 = 0.35$)	Australian Bonds ($w_3 = 0.25$)
Australian Equities	$\sigma_1^2 = 0.025600$	$0.70(0.16)(0.15) = 0.016800$	$0.10(0.16)(0.05) = 0.000800$
Global Equities	0.016800	$\sigma_2^2 = 0.022500$	$0.15(0.15)(0.05) = 0.001125$
Australian Bonds	0.000800	0.001125	$\sigma_3^2 = 0.002500$

1. **Calculate expected portfolio return:**

$$E(r_P) = 0.40(9.50\%) + 0.35(10.50\%) + 0.25(4.50\%) = 3.800\% + 3.675\% + 1.125\% = 8.60\%$$

2. **Calculate portfolio variance $\sigma_P^2 = \sum_i w_i^2 \sigma_i^2 + 2\sum_{i<j} w_i w_j \text{Cov}_{ij}$:**

$$\text{Variance Terms: } (0.40)^2(0.0256) + (0.35)^2(0.0225) + (0.25)^2(0.0025) = 0.004096 + 0.002756 + 0.000156 = 0.007008$$

$$\text{Covariance Terms: } 2(0.40)(0.35)(0.0168) + 2(0.40)(0.25)(0.0008) + 2(0.35)(0.25)(0.001125)$$

$$= 0.004704 + 0.000160 + 0.000197 = 0.005061$$

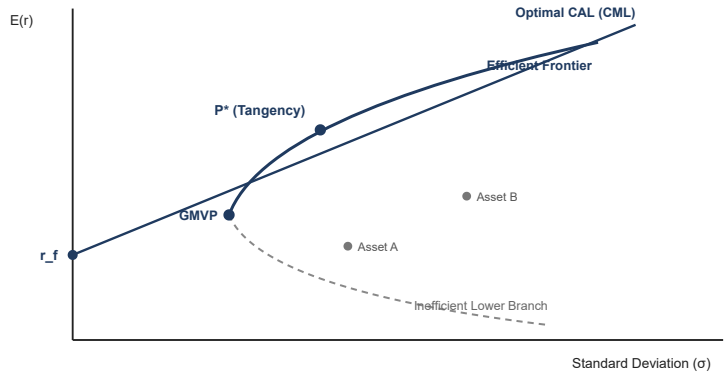
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$$\sigma_P^2 = 0.007008 + 0.005061 = 0.012069 \implies \sigma_P = \sqrt{0.012069} = 10.99\%$$

Compare $\sigma_P = 10.99\%$ against the weighted average standalone volatility $0.40(16\%) + 0.35(15\%) + 0.25(5\%) = 12.90\%$: low cross-asset correlation with bonds trims portfolio risk by 191 basis points.

Introducing a risk-free asset establishes **Tobin's Two-Fund Separation Theorem**: portfolio construction separates cleanly into two independent tasks:

1. **Technical Optimisation (Identical for All Investors):** Security analysts identify the unique tangency portfolio P^* on the efficient frontier that maximises the Sharpe ratio, using purely objective inputs (expected returns, variances, and covariances).
2. **Personal Capital Allocation (Tailored to Risk Aversion):** Each investor combines P^* with the risk-free asset according to their personal risk aversion parameter A, choosing $y^* = \frac{E(r_{P^*}) - r_f}{A\sigma_{P^*}^2}$.



Worked Example 2: Tangency Portfolio P^* and Complete Portfolio Allocation (CBA & RIO)

Continuing Worked Example 1 (CBA: $E(r_1) = 9.00\%$, $\sigma_1 = 16.00\%$; RIO: $E(r_2) = 13.00\%$, $\sigma_2 = 24.00\%$; $\text{Cov}_{12} = 0.0096$), suppose the Australian risk-free rate is $r_f = 4.00\%$. Compute the optimal risky tangency portfolio P^* and determine the overall complete portfolio weights for an investor with risk aversion $A = 3.5$ investing \$250,000.

1. **Calculate expected excess returns:**

$$E(R_1) = 0.0900 - 0.0400 = 0.0500, \qquad E(R_2) = 0.1300 - 0.0400 = 0.0900$$

2. **Compute the tangency portfolio weights w_1^* and w_2^*:**

$$\text{Numerator}_1 = (0.0500)(0.0576) - (0.0900)(0.0096) = 0.002880 - 0.000864 = 0.002016$$

$$\text{Denominator} = (0.0500)(0.0576) + (0.0900)(0.0256) - (0.0500 + 0.0900)(0.0096)$$

$$= 0.002880 + 0.002304 - 0.001344 = 0.003840$$

$$w_1^* = \frac{0.002016}{0.003840} = 0.5250 \text{ (52.50\% CBA)}, \qquad w_2^* = 1 - 0.5250 = 0.4750 \text{ (47.50\% RIO)}$$

3. **Evaluate the expected return, volatility, and Sharpe ratio of P^*:**

$$E(r_{P^*}) = 0.5250(9.00\%) + 0.4750(13.00\%) = 4.725\% + 6.175\% = 10.90\%$$

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$$\sigma_{P^*}^2 = (0.525)^2(0.0256) + (0.475)^2(0.0576) + 2(0.525)(0.475)(0.0096) = 0.007056 + 0.012996 + 0.004788 = 0.024840$$

$$\sigma_{P^*} = \sqrt{0.024840} = 15.76\%$$

$$S_{P^*} = \frac{0.1090 - 0.0400}{0.157607} = 0.4378$$

Note that P^* achieves a higher Sharpe ratio (0.4378) than either CBA alone (0.3125), RIO alone (0.3750), or the GMVP ((10.00% - 4.00%)/14.70% = 0.4082).

4. **Apply Two-Fund Separation for the investor with $A = 3.5$:**

$$y^* = \frac{E(r_{P^*}) - r_f}{A\sigma_{P^*}^2} = \frac{0.0690}{3.5 \times 0.024840} = \frac{0.0690}{0.086940} = 0.79365 \text{ (79.37\% in } P^*\text{, 20.63\% in Treasury Notes)}$$

The final dollar allocations across the \$250,000 portfolio are:

- **Treasury Notes (Risk-Free):** $0.20635 \times \$250,000 = \$51,587$ (20.63%)
- **CBA Shares:** $0.79365 \times 0.5250 \times \$250,000 = \$104,167$ (41.67%)
- **RIO Shares:** $0.79365 \times 0.4750 \times \$250,000 = \$94,246$ (37.70%)

2.3 The Single-Index Model and Treynor-Black Portfolio Construction

Implementing full Markowitz optimisation across N securities requires estimating N expected returns, N variances, and $\frac{N(N-1)}{2}$ unique covariances—a total of $\frac{N(N+3)}{2}$ parameters. For the S&P/ASX 200 ($N = 200$), this demands 20,300 inputs. Worse still, unconstrained mean-variance optimisers act as "error maximisers," assigning extreme long and short weights to small estimation errors in sample covariances and expected returns.

William Sharpe (1963) resolved this dimensionality and estimation bottleneck with the **Single-Index Model (SIM)**, which posits that security co-movement arises primarily from a shared macroeconomic market factor $R_M = r_M - r_f$:

$$R_{i,t} = \alpha_i + \beta_i R_{M,t} + e_{i,t}$$

Where: $R_{i,t} = r_{i,t} - r_f$ is security i's excess return in period t; $R_{M,t} = r_{M,t} - r_f$ is the market index excess return; α_i is the security's expected abnormal return when $R_{M,t} = 0$; β_i is its market sensitivity; and $e_{i,t}$ is a zero-mean firm-specific residual assumed uncorrelated with $R_{M,t}$ and across securities ($\text{Cov}(e_i, e_j) = 0$ for $i \neq j$).

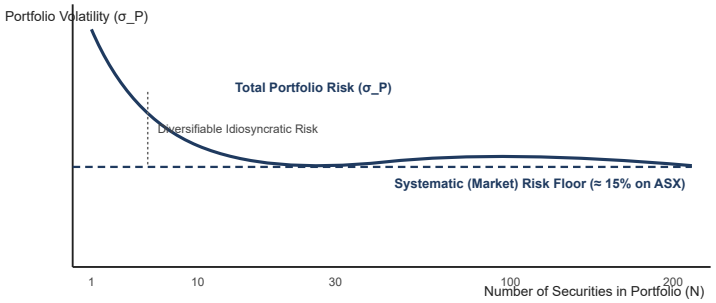
Under the SIM assumptions, only $3N + 2$ inputs are needed ($\alpha_i, \beta_i, \sigma^2(e_i)$ for each of N stocks, plus $E(R_M)$ and σ_M^2)—just 602 parameters for 200 ASX stocks instead of 20,300. Furthermore, total return variance decomposes cleanly into **systematic (market) risk** and **firm-specific (idiosyncratic) risk**, while pairwise covariances simplify to products of betas:

$$\sigma_i^2 = \beta_i^2\sigma_M^2 + \sigma^2(e_i), \qquad \text{Cov}(r_i, r_j) = \beta_i\beta_j\sigma_M^2 \quad (i \neq j)$$

Where: $\beta_i^2\sigma_M^2$ is systematic variance; $\sigma^2(e_i)$ is firm-specific variance; and σ_M^2 is the variance of the market index.

As portfolio size N increases with equal weights $1/N$, systematic variance remains $\bar{\beta}^2\sigma_M^2$, whereas idiosyncratic variance $\frac{1}{N^2}\sum\sigma^2(e_i) = \frac{\bar{\sigma}^2(e)}{N}$ shrinks toward zero. Empirical studies on the ASX demonstrate that holding 25 to 35 well-dispersed stocks across sectors eliminates the vast majority of diversifiable idiosyncratic risk.

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Worked Example 2B: Single-Index Variance Decomposition and Covariance Estimation

Suppose the S&P/ASX 200 index has annual volatility $\sigma_M = 15.00\%$ ($\sigma_M^2 = 0.0225$). Single-index regressions produce $\beta_{\text{WDS}} = 1.20$ and $\sigma(e_{\text{WDS}}) = 22.00\%$ for Woodside Energy (WDS), and $\beta_{\text{WOW}} = 0.60$ and $\sigma(e_{\text{WOW}}) = 14.00\%$ for Woolworths Group (WOW). Decompose each stock's variance and compute their implied covariance and correlation.

1. **Decompose WDS and WOW total variance into systematic and idiosyncratic components:**

$$\sigma^2_{\text{WDS}} = (1.20)^2(0.0225) + (0.22)^2 = 0.0324 + 0.0484 = 0.0808 \implies \sigma_{\text{WDS}} = 28.43\%, \quad R^2_{\text{WDS}} = \frac{0.0324}{0.0808} = 40.10\%$$

$$\sigma^2_{\text{WOW}} = (0.60)^2(0.0225) + (0.14)^2 = 0.0081 + 0.0196 = 0.0277 \implies \sigma_{\text{WOW}} = 16.64\%, \quad R^2_{\text{WOW}} = \frac{0.0081}{0.0277} = 29.24\%$$

2. **Estimate SIM covariance and correlation between WDS and WOW:**

$$\text{Cov}(r_{\text{WDS}}, r_{\text{WOW}}) = \beta_{\text{WDS}}\beta_{\text{WOW}}\sigma_M^2 = (1.20)(0.60)(0.0225) = 0.0162$$

$$\rho_{\text{WDS,WOW}} = \frac{0.0162}{0.28425 \times 0.16643} = 0.3424$$

The Treynor-Black Active-Passive Portfolio Model

Institutional managers frequently combine a core passive market index fund M with an active "satellite" portfolio A of mispriced stocks identified via security analysis ($\alpha_i \neq 0$). The **Treynor-Black (1973)** model solves for the exact optimal weights that maximise the overall Sharpe ratio while balancing alpha generation against idiosyncratic tracking risk:

1. **Construct the Optimal Active Portfolio A:** Assign initial unadjusted active weights proportional to each stock's alpha-to-idiosyncratic-variance ratio, $w_i^0 = \frac{\alpha_i}{\sigma^2(e_i)}$, and normalise so active weights sum to 1: $w_i^A = \frac{w_i^0}{\sum_{j=1}^{k} w_j^0}$. Compute active portfolio properties $\alpha_A = \sum w_i^A \alpha_i$, $\beta_A = \sum w_i^A \beta_i$, and $\sigma^2(e_A) = \sum (w_i^A)^2 \sigma^2(e_i)$.
2. **Determine the Optimal Allocation to Active Portfolio A vs Passive Index M:**

$$w_A^0 = \frac{\alpha_A/\sigma^2(e_A)}{E(R_M)/\sigma_M^2}, \qquad w_A^* = \frac{w_A^0}{1 + (1 - \beta_A)w_A^0}, \qquad w_M^* = 1 - w_A^*$$

   Where: w_A^0 is the unadjusted active allocation assuming $\beta_A = 1$; w_A^* adjusts for active beta exposure relative to 1.0; and w_M^* is the weight placed in the passive market index.
3. **Evaluate the Combined Sharpe Ratio Improvement:** The squared Sharpe ratio of the optimal combined risky portfolio P equals the squared market Sharpe ratio plus the squared **Information Ratio** ($IR_A = \alpha_A/\sigma(e_A)$) of the active portfolio:

$$S_P^2 = S_M^2 + \left[\frac{\alpha_A}{\sigma(e_A)}\right]^2 = S_M^2 + \sum_{i=1}^{k} \left[\frac{\alpha_i}{\sigma(e_i)}\right]^2$$

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Worked Example 3: Treynor-Black Active-Passive Allocation on the ASX

A fund manager holds a passive S&P/ASX 200 index fund M with expected excess return $E(R_M) = 6.00\%$ and volatility $\sigma_M = 15.00\%$ (hence $S_M = 0.4000$). Fundamental equity research identifies two undervalued ASX stocks: Stock X ($\alpha_X = +3.00\%, \beta_X = 1.20, \sigma(e_X) = 20.00\%$) and Stock Y ($\alpha_Y = +1.28\%, \beta_Y = 0.80, \sigma(e_Y) = 16.00\%$). Construct the optimal active-passive portfolio.

1. **Compute unadjusted active weights and normalise to form Active Portfolio A:**

$$w_X^0 = \frac{0.0300}{(0.20)^2} = \frac{0.0300}{0.0400} = 0.7500, \qquad w_Y^0 = \frac{0.0128}{(0.16)^2} = \frac{0.0128}{0.0256} = 0.5000$$

$$w_X^A = \frac{0.7500}{0.7500 + 0.5000} = 0.6000\ (60.0\%), \qquad w_Y^A = \frac{0.5000}{1.2500} = 0.4000\ (40.0\%)$$

2. **Compute Active Portfolio A parameters and Information Ratio:**

$$\alpha_A = 0.60(3.00\%) + 0.40(1.28\%) = 2.312\%, \qquad \beta_A = 0.60(1.20) + 0.40(0.80) = 1.040$$

$$\sigma^2(e_A) = (0.60)^2(0.0400) + (0.40)^2(0.0256) = 0.014400 + 0.004096 = 0.018496 \implies \sigma(e_A) = 13.60\%$$

$$IR_A = \frac{0.02312}{0.1360} = 0.1700$$

3. **Compute the optimal split between Active Portfolio A and Passive Index M:**

$$w_A^0 = \frac{0.02312/0.018496}{0.0600/0.0225} = \frac{1.2500}{2.6667} = 0.46875$$

$$w_A^* = \frac{0.46875}{1 + (1 - 1.040)(0.46875)} = \frac{0.46875}{0.98125} = 0.4777\ (47.77\%\ \text{Active}, 52.23\%\ \text{Passive Index})$$

Thus, total portfolio weights are $0.4777 \times 0.60 = 28.66\%$ in Stock X, $0.4777 \times 0.40 = 19.11\%$ in Stock Y, and 52.23% in the S&P/ASX 200 index.

4. **Verify the combined portfolio Sharpe ratio:**

$$S_P = \sqrt{(0.4000)^2 + (0.1700)^2} = \sqrt{0.1600 + 0.0289} = \sqrt{0.1889} = 0.4346$$

2.4 The Capital Asset Pricing Model (CAPM), SML, and Extensions

While Markowitz portfolio theory prescribes how a single investor should optimise, the **Capital Asset Pricing Model (CAPM)** developed independently by Sharpe (1964), Lintner (1965), and Mossin (1966) derives the general equilibrium pricing of risky assets when *all* investors optimise simultaneously. Under classical assumptions—mean-variance investors with identical one-period horizons, homogeneous expectations, frictionless markets without taxes or transaction costs, unrestricted borrowing/lending at r_f, and all assets publicly traded—every investor holds the same tangency portfolio, which in market-clearing equilibrium must be the value-weighted **Market Portfolio** M.

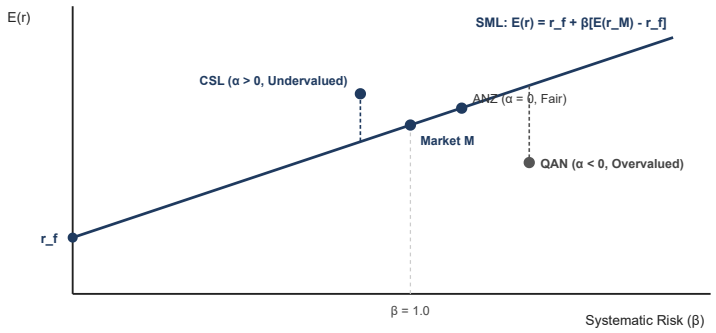
The Capital Allocation Line passing through M is termed the **Capital Market Line (CML)**, describing efficient portfolios: $E(r_P) = r_f + \frac{E(r_M) - r_f}{\sigma_M}\sigma_P$. For any individual security or arbitrary portfolio i, however, its risk contribution to M is measured solely by its covariance with the market, scaled by market variance—its **beta** (β_i):

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$$E(r_i) = r_f + \beta_i \left[E(r_M) - r_f\right], \qquad \beta_i = \frac{\text{Cov}(r_i, r_M)}{\sigma_M^2}$$

Where: $E(r_i)$ is the CAPM equilibrium expected return on asset i; r_f is the risk-free rate; $E(r_M) - r_f$ is the Market Equity Risk Premium (ERP); and β_i measures asset i's systematic risk relative to the market portfolio.

Graphing $E(r_i)$ against β_i yields the **Security Market Line (SML)**. Crucially, the CML applies *only* to efficient portfolios lying on the frontier (measuring risk by total σ_P), whereas the SML prices *all* individual securities and inefficient portfolios in equilibrium (measuring risk by systematic β_i).



Feature	Capital Market Line (CML)	Security Market Line (SML)
Risk Measure (Horizontal Axis)	Total volatility (σ_P)	Systematic beta (β_i)
Slope	Market Sharpe Ratio: $\frac{E(r_M)-r_f}{\sigma_M}$	Market Risk Premium: $E(r_M) - r_f$
Applicable Universe	Efficient portfolios combining r_f and Market M	All individual securities, inefficient & efficient portfolios
Implication for Inefficient Assets	Plot strictly below the CML	Plot directly on the SML in equilibrium (zero alpha)

Alpha Analysis, CAPM Extensions, and Roll's Critique

If a security's forecast expected return $E_{\text{forecast}}(r_i)$ deviates from its CAPM equilibrium return, the discrepancy is **Jensen's alpha** ($\alpha_i = E_{\text{forecast}}(r_i) - [r_f + \beta_i(E(r_M) - r_f)]$). Securities with $\alpha_i > 0$ plot *above* the SML and are undervalued (buy candidates), while securities with $\alpha_i < 0$ plot *below* the SML and are overvalued.

Relaxing strict CAPM assumptions yields important theoretical extensions:

- **Black's Zero-Beta CAPM (1972):** When risk-free borrowing is restricted or unavailable, investors replace r_f with the expected return on a minimum-variance portfolio Z uncorrelated with M ($\text{Cov}(r_Z, r_M) = 0$), yielding $E(r_i) = E(r_Z) + \beta_i[E(r_M) - E(r_Z)]$ with a flatter SML slope.
- **Consumption CAPM (Breeden, 1979):** Replaces market-return beta with consumption beta, pricing assets based on their covariance with aggregate real consumption growth—explaining why assets that pay off poorly during recessions command higher risk premiums.
- **Liquidity-Adjusted CAPM (Acharya & Pedersen, 2005):** Recognises that investors require a return premium not only for market beta but also for illiquidity level and illiquidity-covariance risk ("flight to liquidity" during crises).
- **Roll's Critique (1977):** Richard Roll proved mathematically that the CAPM's sole testable implication is that the true market portfolio is mean-variance efficient. Because the true global market portfolio—encompassing human capital, private businesses, residential property, and foreign assets—is unobservable, empirical tests using narrow proxies like the S&P/ASX 200 test only the efficiency of the proxy rather than invalidating the underlying theory.

Worked Example 4: SML Valuation and Market-Neutral Alpha Construction

Assume the Australian 10-year risk-free proxy is $r_f = 4.00\%$ and the expected market return on the S&P/ASX 200 is $E(r_M) = 10.00\%$ (ERP = 6.00%). Analyst consensus forecasts the following 1-year returns and betas for three ASX stocks: CSL Limited ($\beta = 0.85$, forecast 10.60%), ANZ Group ($\beta = 1.15$, forecast 10.90%), and Qantas Airways ($\beta = 1.35$, forecast 10.50%). Evaluate each stock against the SML.

ASX Stock	Beta (β_i)	CAPM Required Return	Analyst Forecast	Alpha (α_i)	SML Position & Valuation
CSL Limited	0.85	$4.00\% + 0.85(6.00\%) = 9.10\%$	10.60%	+1.50%	Above SML — Undervalued (Buy)
ANZ Group	1.15	$4.00\% + 1.15(6.00\%) = 10.90\%$	10.90%	0.00%	On SML — Fairly Priced (Hold)

ASX Stock	Beta (β_i)	CAPM Required Return	Analyst Forecast	Alpha (α_i)	SML Position & Valuation
Qantas Airways	1.35	$4.00\% + 1.35(6.00\%) = 12.10\%$	10.50%	-1.60%	Below SML — Overvalued (Sell/Short)

An active manager can isolate CSL's +1.50% alpha while eliminating systematic market exposure by investing \$1,000,000 in CSL ($\beta = 0.85$) and simultaneously shorting \$850,000 of SPI 200 index futures (or index ETF, $\beta = 1.00$), creating a zero-beta portfolio with net systematic exposure $1.00(0.85) - 0.85(1.00) = 0$.

Empirical Testing of CAPM and Beta Estimation in Practice

Testing whether the SML holds empirically presents an econometric challenge because individual stock betas are estimated with measurement error, inducing attenuation bias in cross-sectional regressions. Black, Jensen, and Scholes (1972) and Fama and MacBeth (1973) overcame this errors-in-variables problem using a **two-pass portfolio regression**: first-pass time-series regressions estimate betas for diversified beta-sorted portfolios, and second-pass cross-sectional regressions test whether average returns are linearly related to portfolio beta:

$$\bar{r}_p - r_f = \gamma_0 + \gamma_1 \hat{\beta}_p + \gamma_2 \hat{\beta}_p^2 + \gamma_3 \hat{\sigma}^2(e_p) + \eta_p$$

Where: $\bar{r}_p - r_f$ is portfolio p's mean excess return; $\hat{\beta}_p$ is its estimated market beta; γ_0 tests whether the intercept equals zero; γ_1 tests whether the market price of beta risk is positive; γ_2 tests linearity; and γ_3 tests whether idiosyncratic risk is unpriced.

Classic Fama-MacBeth tests confirmed linearity ($\gamma_2 \approx 0$) and confirmed that idiosyncratic variance is unpriced ($\gamma_3 \approx 0$), but consistently found an empirical SML that is *too flat* ($\gamma_0 > 0$ and $\gamma_1 < \bar{r}_M - r_f$)—consistent with Black's Zero-Beta CAPM and the empirical outperformance of low-beta ("betting against beta") equities.

When estimating forward-looking betas for ASX equities, practitioners typically regress 60 months of excess stock returns against the S&P/ASX 300 Accumulation Index. Because historical OLS betas exhibit mean reversion toward the market average of 1.0 as firms mature and diversify, financial data providers (such as Bloomberg and S&P Capital IQ) apply **Blume's Shrinkage Adjustment** (or Bayesian Vasicek shrinkage) to improve out-of-sample forecast accuracy:

$$\beta_{\text{adjusted}} = \frac{2}{3}\hat{\beta}_{\text{OLS}} + \frac{1}{3}(1.0)$$

Where: $\hat{\beta}_{\text{OLS}}$ is the raw historical regression beta; and β_{adjusted} shrinks extreme estimates one-third of the way toward the grand mean of 1.0.

2.5 Arbitrage Pricing Theory (APT) and Multi-Factor Models

Stephen Ross (1976) developed **Arbitrage Pricing Theory (APT)** as a more general alternative to CAPM that eliminates restrictive utility assumptions and the unobservable market portfolio requirement. APT rests on three propositions: security returns are generated by a K-factor linear data-generating process, sufficient securities exist to form well-diversified portfolios with negligible idiosyncratic risk, and competitive markets do not permit persistent riskless arbitrage profits.

Under a K-factor model, the return on asset i responds to unexpected macroeconomic factor surprises $F_1, F_2, \ldots, F_K$ (each having zero expected value $E(F_k) = 0$):

$$r_i = E(r_i) + \sum_{k=1}^{K} \beta_{ik} F_k + e_i$$

Where: $E(r_i)$ is asset i's expected return; β_{ik} is its sensitivity (factor loading) to surprise factor F_k; and e_i is an idiosyncratic shock uncorrelated across assets.

For any **well-diversified portfolio** P, idiosyncratic variance $\sigma^2(e_P)$ approaches zero. By the Law of One Price, two well-diversified portfolios sharing identical factor sensitivities β_{Pk} must offer identical expected returns; otherwise an investor could buy the higher-return portfolio and short the lower-return portfolio for a guaranteed risk-free profit with zero net capital. No-arbitrage therefore dictates a linear factor pricing equation:

$$E(r_i) = r_f + \sum_{k=1}^{K} \beta_{ik}\lambda_k = r_f + \beta_{i1}\lambda_1 + \beta_{i2}\lambda_2 + \cdots + \beta_{iK}\lambda_K$$

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Where: $\lambda_k = E(r_{Fk}) - r_f$ is the factor risk premium (price of risk) earned by a pure unit-sensitivity factor portfolio with $\beta_k = 1$ and all other factor betas equal to zero.

Worked Example 5: Constructing a Two-Factor APT Arbitrage Trade

Suppose Australian asset returns are governed by two macroeconomic factors: Factor 1 (unexpected domestic GDP growth, $\lambda_1 = 5.00\%$) and Factor 2 (unexpected CPI inflation shock, $\lambda_2 = -2.00\%$), with risk-free rate $r_f = 4.00\%$. Benchmark pure-factor portfolios F_1 ($\beta_1 = 1, \beta_2 = 0$) and F_2 ($\beta_1 = 0, \beta_2 = 1$) trade fairly at expected returns of $4.00\% + 5.00\% = 9.00\%$ and $4.00\% - 2.00\% = 2.00\%$, respectively. A well-diversified portfolio P with $\beta_{P1} = 1.20$ and $\beta_{P2} = 0.50$ currently offers an expected return of 10.50%. Identify the mispricing and construct a $1,000,000 zero-investment, zero-factor-risk arbitrage strategy.

1. **Compute the fair APT expected return for Portfolio P:**

$$E(r_P)_{\text{APT}} = 4.00\% + 1.20(5.00\%) + 0.50(-2.00\%) = 4.00\% + 6.00\% - 1.00\% = 9.00\%$$

   Because P promises 10.50% versus a fair APT return of 9.00%, P is undervalued by +1.50% (+150 bps).

2. **Build a replicating portfolio R matching P's factor betas using F_1, F_2, and r_f:** Place weight 1.20 in F_1, weight 0.50 in F_2, and the balancing weight $1 - 1.20 - 0.50 = -0.70$ (borrowing) in r_f. Verify factor betas and expected return:

$$\beta_{R1} = 1.20(1) + 0.50(0) = 1.20, \qquad \beta_{R2} = 1.20(0) + 0.50(1) = 0.50$$

$$E(r_R) = 1.20(9.00\%) + 0.50(2.00\%) - 0.70(4.00\%) = 10.80\% + 1.00\% - 2.80\% = 9.00\%$$

3. **Execute the arbitrage (Long $1,000,000 P, Short $1,000,000 R):** Buy $1,000,000 of Portfolio P, short-sell $1,200,000 of F_1, short-sell $500,000 of F_2, and lend $700,000 at r_f. Net cash outlay is $\$1,000,000 - \$1,200,000 - \$500,000 + \$700,000 = \$0$. Net factor betas are $\beta_1 = 1.20 - 1.20 = 0$ and $\beta_2 = 0.50 - 0.50 = 0$. Because P is well-diversified ($\sigma(e_P) \approx 0$), the strategy locks in an expected riskless profit of $\$1,000,000 \times (10.50\% - 9.00\%) = \$15,000$. Arbitrageurs executing this trade bid up P's price until its expected return compresses to 9.00%.

Empirical Anomalies and Multi-Factor Asset Pricing Models

Empirical tests from the 1980s onward uncovered systematic cross-sectional return patterns—**empirical anomalies**—that single-beta CAPM could not explain: small-capitalisation firms outperformed large-cap firms after adjusting for beta (the *size effect*, Banz 1981); high book-to-market ratio ("value") stocks outperformed low book-to-market ("growth") stocks; and past 12-month winners continued outperforming past losers over 3- to 12-month horizons (*momentum*, Jegadeesh and Titman 1993).

To capture these systematic return premia, Eugene Fama and Kenneth French (1993, 2015) and Mark Carhart (1997) developed multi-factor characteristic-sorted factor models. The **Fama-French Five-Factor Model** adds profitability and corporate investment factors to the classic three-factor specification (market, size, and value):

$$E(r_i) - r_f = \beta_{iM}\,[E(r_M) - r_f] + \beta_{i,\text{SMB}} E(\text{SMB}) + \beta_{i,\text{HML}} E(\text{HML}) + \beta_{i,\text{RMW}} E(\text{RMW}) + \beta_{i,\text{CMA}} E(\text{CMA})$$

Where: $E(r_M) - r_f$ is the market risk premium; SMB (Small Minus Big) is the size factor return spread between small-cap and large-cap portfolios; HML (High Minus Low) is the value factor spread between high and low book-to-market stocks; RMW (Robust Minus Weak) captures operating profitability; and CMA (Conservative Minus Aggressive) captures low-investment versus high-investment asset growth. The Carhart 4-factor model replaces RMW and CMA in the 3-factor model with a momentum factor WML (Winners Minus Losers).

Factor	Portfolio Construction	Risk-Based Explanation	Australian (ASX) Empirical Evidence
Market (MKT)	Value-weighted market less 30/90-day risk-free rate	Compensation for aggregate macroeconomic and equity market risk	Positive long-run ERP (~6.0%–7.0% p.a. including franking credits)
Size (SMB)	Small-cap portfolio minus large-cap portfolio	Lower liquidity, higher distress risk, narrower analyst coverage	Concentrated outside S&P/ASX 100 in ASX Small Ordinaries & micro-caps
Value (HML)	High book-to-market minus low book-to-market stocks	Distress risk and irreversible capital rigidity in downturns	Strong historical premium across ASX banks, insurers, and industrials
Momentum (WML)	Prior 12-2 month winners minus prior 12-2 month losers	Investor underreaction, delayed news diffusion, or crash risk	Statistically robust on ASX, though higher turnover erodes net returns

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Factor	Portfolio Construction	Risk-Based Explanation	Australian (ASX) Empirical Evidence
Profitability (RMW) & Investment (CMA)	High operating profit minus weak; low asset growth minus high	Q-theory of investment: higher expected cash flows imply higher discount rates	Helps explain why unprofitable speculative ASX miners underperform

Worked Example 6: Estimating Cost of Equity via the Fama-French 3-Factor Model

An Australian equity strategist estimates the cost of equity for an ASX mid-cap value industrial firm using Fama-French 3-factor loadings estimated from 60 months of excess returns: $\beta_M = 1.10$, $\beta_{\text{SMB}} = 0.65$, and $\beta_{\text{HML}} = 0.50$. Current forward factor premiums for the Australian market are $r_f = 4.00\%$, $E(r_M) - r_f = 6.00\%$, $E(\text{SMB}) = 2.50\%$, and $E(\text{HML}) = 3.20\%$.

1. **Compare against single-factor CAPM:**

$$E(r_i)_{\text{CAPM}} = 4.00\% + 1.10(6.00\%) = 10.60\%$$

2. **Compute the Fama-French 3-Factor expected return:**

$$E(r_i)_{\text{FF3}} = 4.00\% + 1.10(6.00\%) + 0.65(2.50\%) + 0.50(3.20\%)$$

$$= 4.00\% + 6.60\% + 1.625\% + 1.60\% = 13.825\% \approx 13.83\%$$

Because the firm exhibits positive exposure to both small-cap risk ($+1.63\%$) and value distress risk ($+1.60\%$), its multi-factor required return exceeds CAPM by 323 basis points.

Factor Investing, Smart Beta ETFs, and Momentum Crash Risk

Modern institutional asset managers operationalise multi-factor pricing through transparent, rules-based **Smart Beta (factor) ETFs** and multi-factor quantitative equity mandates. Rather than paying active-management fees for stock picking that merely repackages size, value, quality, or low-volatility tilts, superannuation funds harvest these documented factor premia systematically at low cost. However, multi-factor investing introduces distinct risks that must be managed at the total portfolio level:

- **Factor Cyclicality and Crowding:** Individual factors experience multi-year drawdowns—for instance, the value factor (HML) suffered severe underperformance from 2017 to 2020 before rebounding sharply during the 2021–2022 inflation cycle. Combining negatively or weakly correlated factors (such as Value and Momentum, which exhibit negative correlation on the ASX) smooths active performance across macroeconomic regimes.
- **Momentum Crash Risk:** Following sharp market troughs (such as March–April 2009 or late 2020 vaccine announcements), heavily shorted "loser" stocks often rebound violently while prior defensive "winner" stocks lag, inflicting abrupt left-tail losses on momentum strategies unless volatility-scaled.
- **Data-Mining vs True Risk Premia:** Over 300 candidate return predictors have been proposed in academic literature ("the factor zoo"). Rigorous asset pricing demands out-of-sample replication across international markets including Australia, economic rationale (either systematic risk compensation or persistent behavioural limits to arbitrage), and net-of-transaction-cost survivability.

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