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Bachelor of Commerce (B2001) | Level 2 • Part B Major | Monash University

Derivatives

Course Code: BFC2751 • Module 4 Notes

Module 4: Black-Scholes-Merton Model, the Greeks, Implied Volatility & Risk Management

4.1 Stochastic Calculus Foundations and the Black-Scholes-Merton PDE

In Module 3, asset price dynamics were modelled over discrete time steps on a binomial lattice. As the step duration $\Delta t \to 0$, the binomial distribution converges to a continuous-time diffusion process governed by **Geometric Brownian Motion (GBM)**, the cornerstone of the Black-Scholes-Merton (BSM) framework (Black and Scholes, 1973; Merton, 1973).

Wiener Processes, Geometric Brownian Motion and Itô's Lemma

A variable z follows a standard **Wiener process** (standard Brownian motion) if over any infinitesimal interval dt, its increment dz satisfies $dz = \epsilon \sqrt{dt}$, where $\epsilon \sim \mathrm{N}(0,1)$ is a standard normal random variable, and increments across non-overlapping intervals are statistically independent. Consequently, $E[dz] = 0$, $\mathrm{Var}(dz) = dt$, and the change over a finite horizon T satisfies $z(T) - z(0) \sim \mathrm{N}(0, T)$.

Because share prices cannot become negative and returns scale proportionally with price level, equity prices S_t are modelled as a Geometric Brownian Motion:

$$\frac{dS}{S} = \mu\, dt + \sigma\, dz \quad \Leftrightarrow \quad dS = \mu S\, dt + \sigma S\, dz$$

Where: S is the underlying asset price; μ is the expected instantaneous annualised rate of return (drift); σ is the instantaneous annualised volatility of the asset's return; and dz is an increment of a standard Wiener process.

Because $dz^2 = dt$ in quadratic variation (while higher-order terms dt^2 and $dt\, dz$ vanish), ordinary chain-rule calculus does not apply to stochastic functions $G(S,t)$. Instead, **Itô's Lemma** provides the stochastic differential equation for any twice-differentiable function $G(S,t)$:

$$dG = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right) dt + \frac{\partial G}{\partial S}\sigma S\, dz$$

Applying Itô's Lemma to $G(S) = \ln S$ (for which $\frac{\partial G}{\partial S} = \frac{1}{S}$, $\frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}$, and $\frac{\partial G}{\partial t} = 0$) shows that $\ln S$ follows a constant-parameter arithmetic Brownian motion:

$$d(\ln S) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\, dz \implies \ln S_T \sim \mathrm{N}\left(\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right) T,\ \sigma^2 T\right)$$

Where: $\ln S_T$ is normally distributed, meaning the terminal share price S_T follows a **log-normal distribution**; and $-\frac{\sigma^2}{2}$ is the volatility drag (Itô correction) separating the arithmetic instantaneous return μ from the expected continuously compounded growth rate.

Properties of the Log-Normal Stock Price Distribution

From the normality of $\ln S_T$, two essential properties follow for risk analysis and option payoff integration. First, the expected future share price is $E[S_T] = S_0 e^{\mu T}$, whereas the *median* share price is *S_0 e^{(mu - sigma^2/2)T} < E[S_T]* due to right-skewness. Second, the 95% confidence interval for S_T over horizon T is bounded by:

$$S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) T - 1.96\,\sigma \sqrt{T}\right] \le S_T \le S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) T + 1.96\,\sigma \sqrt{T}\right]$$

Where: μ is the expected annualised return; σ is annualised volatility; and 1.96 is the two-tailed 95% standard normal critical quantile.

Derivation of the Black-Scholes-Merton Partial Differential Equation

Black, Scholes, and Merton assumed frictionless markets: continuous trading with zero transaction costs, constant risk-free rate r and constant volatility σ, unlimited short selling, no dividends (relaxed by Merton), and log-normal asset prices. Under these conditions, consider a derivative whose price $f(S,t)$ satisfies Itô's Lemma. Constructing a continuously rebalanced portfolio Π consisting of short one derivative and long $\frac{\partial f}{\partial S}$ shares gives:

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$$\Pi = -f + \frac{\partial f}{\partial S}S, \qquad d\Pi = -df + \frac{\partial f}{\partial S}dS = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt$$

Notice that the stochastic term $\sigma S\frac{\partial f}{\partial S}dz$ cancels out identically. Because $d\Pi$ contains no dz uncertainty over the instant dt, the portfolio Π is instantaneously risk-free and must earn the risk-free rate r to preclude arbitrage ($d\Pi = r\Pi\, dt$). Substituting Π and $d\Pi$ yields the celebrated **Black-Scholes-Merton Partial Differential Equation (PDE)**:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = rf$$

Where: f is the price of any derivative security contingent on S and t; r is the continuously compounded risk-free rate; and σ is the underlying asset volatility. Remarkably, the stock's expected return μ vanishes completely from the PDE, establishing risk-neutral valuation in continuous time.

4.2 The Black-Scholes-Merton Pricing Formulae and Extensions

Solving the BSM PDE subject to the European call terminal boundary condition $c(S_T, T) = \max(S_T - K, 0)$ and European put boundary condition $p(S_T, T) = \max(K - S_T, 0)$ produces closed-form pricing equations. Incorporating Merton's extension for a continuous proportional dividend yield q per annum (where $q = 0$ recovers the classic non-dividend Black-Scholes formula), the European call price c and put price p are:

$$c = S_0 e^{-qT}N(d_1) - Ke^{-rT}N(d_2), \qquad p = Ke^{-rT}N(-d_2) - S_0 e^{-qT}N(-d_1)$$

$$d_1 = \frac{\ln(S_0/K) + (r - q + \frac{\sigma^2}{2})\,T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}$$

Where: S_0 is the current spot share price; K is the option strike price; T is time to maturity in years; r is the continuously compounded risk-free rate; q is the continuously compounded annual dividend yield; σ is the annualised return volatility; and $N(\cdot)$ is the standard normal cumulative distribution function.

In economic terms, $N(d_2)$ equals the **risk-neutral probability** that a European call finishes in-the-money ($S_T > K$), so $Ke^{-rT}N(d_2)$ is the present value of the expected strike payment upon exercise. Meanwhile, $S_0 e^{-qT}N(d_1)$ represents the present value of receiving the share conditional on $S_T > K$ under the risk-neutral measure, with $e^{-qT}N(d_1)$ serving as the option's spot Delta.

Discrete Dividends, FX Options (Garman-Kohlhagen) and Futures Options (Black 1976)

- **Discrete Known Dividends (Escrowed Dividend Model):** When an ASX stock pays known discrete cash dividends D_j at ex-dividend dates $t_j \le T$, replace $S_0 e^{-qT}$ with $S_0^* = S_0 - \sum_j D_j e^{-rt_j}$ (the spot price net of the present value of dividends) and set $q = 0$ inside d_1 and d_2.
- **Foreign Exchange Options (Garman-Kohlhagen Model):** Setting $q = r_f$ (the foreign continuously compounded risk-free rate) and $r = r_d$ (the domestic risk-free rate) values a European currency option where S_0 is quoted in domestic currency per unit of foreign currency:

  $$c_{\text{FX}} = S_0 e^{-r_f T}N(d_1) - Ke^{-r_d T}N(d_2), \qquad p_{\text{FX}} = Ke^{-r_d T}N(-d_2) - S_0 e^{-r_f T}N(-d_1)$$

- **Options on Futures (Black's 1976 Model):** Because a futures position requires zero initial cash outlay, its risk-neutral drift is zero (equivalent to setting $q = r$ and substituting $F_0 e^{-rT} = S_0 e^{-qT}$):

  $$c_{\text{fut}} = e^{-rT}\left[F_0 N(d_1) - KN(d_2)\right], \qquad d_1 = \frac{\ln(F_0/K) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}$$

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Worked Example 1: Merton BSM Pricing on an ASX Dividend-Paying Equity

Shares of an ASX-listed healthcare company trade at $S_0 = \$52.00$, paying a continuous dividend yield of $q = 3.20\%$ p.a. The continuously compounded risk-free rate is $r = 4.20\%$ p.a., and annualised volatility is $\sigma = 26.0\%$. Price 6-month ($T = 0.50$ years) European call and put options with strike $K = \$50.00$, and verify your results via Put-Call Parity.

1. **Calculate d_1 and d_2:**

$$\sigma \sqrt{T} = 0.26 \times \sqrt{0.50} = 0.26 \times 0.707107 = 0.183848$$

$$d_1 = \frac{\ln(52/50) + (0.042 - 0.032 + \frac{0.26^2}{2}) \times 0.50}{0.183848} = \frac{0.039221 + (0.010 + 0.0338) \times 0.50}{0.183848} = \frac{0.061121}{0.183848} = 0.3325$$

$$d_2 = 0.3325 - 0.183848 = 0.1486$$

2. **Evaluate standard normal CDF values:**

$$N(d_1) = N(0.3325) = 0.6302, \qquad N(d_2) = N(0.1486) = 0.5591$$

$$N(-d_1) = 1 - 0.6302 = 0.3698, \qquad N(-d_2) = 1 - 0.5591 = 0.4409$$

3. **Compute European Call and Put prices:** First compute discounted spot and strike terms:

$$S_0 e^{-qT} = 52.00\, e^{-0.032 \times 0.50} = 52.00 \times 0.984127 = \$51.1746$$

$$K e^{-rT} = 50.00\, e^{-0.042 \times 0.50} = 50.00 \times 0.979219 = \$48.9609$$

$$c = (\$51.1746 \times 0.6302) - (\$48.9609 \times 0.5591) = \$32.2524 - \$27.3733 = \$4.8791$$

$$p = (\$48.9609 \times 0.4409) - (\$51.1746 \times 0.3698) = \$21.5876 - \$18.9222 = \$2.6654$$

4. **Verify via Put-Call Parity:**

$$c + K e^{-rT} = \$4.8791 + \$48.9609 = \$53.8400, \qquad p + S_0 e^{-qT} = \$2.6654 + \$51.1746 = \$53.8400$$

Both sides equal A$53.8400 identically.

Worked Example 2: Valuing an AUD/USD Foreign Exchange Call Option (Garman-Kohlhagen)

An Australian importer requires a 3-month ($T = 0.25$ years) European call option to buy $\text{US}\$1,000,000$ at a strike exchange rate of $K = 1.5500$ AUD/USD. The current spot rate is $S_0 = 1.5400$ AUD/USD, annualised FX volatility is $\sigma = 11.0\%$, and continuously compounded 3-month risk-free rates are $r_{\text{AUD}} = 4.30\%$ and $r_{\text{USD}} = 4.80\%$. Compute the total AUD option premium.

1. **Compute d_1 and d_2:**

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$$\sigma \sqrt{T} = 0.11 \times \sqrt{0.25} = 0.0550$$

$$d_1 = \frac{\ln(1.5400/1.5500) + (0.0430 - 0.0480 + \frac{0.11^2}{2}) \times 0.25}{0.0550} = \frac{-0.006473 + 0.0002625}{0.0550} = -0.1129$$

$$d_2 = -0.1129 - 0.0550 = -0.1679$$

2. **Evaluate normal probabilities and unit option value:** $N(-0.1129) = 0.4551$ and $N(-0.1679) = 0.4333$.

$$c_{\text{FX}} = 1.5400\, e^{-0.048 \times 0.25}(0.4551) - 1.5500\, e^{-0.043 \times 0.25}(0.4333) = 0.692498 - 0.664554 = 0.027944 \text{ AUD per USD}$$

3. **Total AUD Premium for US$1,000,000 Notional:**

$$\text{Total Premium} = 1{,}000{,}000 \times 0.027944 = \$27{,}944.04 \text{ AUD}$$

Worked Example 3: Valuing an ASX SPI 200 Futures Option via Black's (1976) Model

Three-month ASX SPI 200 index futures currently trade at $F_0 = 7{,}850$ points. The continuously compounded risk-free rate is $r = 4.20\%$ p.a., and implied volatility of the futures price is $\sigma = 18.0\%$ p.a. Using Black's (1976) futures option model, calculate the fair 3-month ($T = 0.25$ years) European call and put prices (in index points) at strike $K = 7{,}900$.

1. **Compute d_1 and d_2:**

$$\sigma \sqrt{T} = 0.18 \times \sqrt{0.25} = 0.0900$$

$$d_1 = \frac{\ln(7{,}850/7{,}900) + \frac{0.18^2}{2} \times 0.25}{0.0900} = \frac{-0.006349 + 0.004050}{0.0900} = -0.0255, \qquad d_2 = -0.0255 - 0.0900 = -0.1155$$

2. **Evaluate normal CDFs and compute European futures call and put prices:** $N(-0.0255) = 0.4898$, $N(-0.1155) = 0.4540$, $N(-d_1) = 0.5102$, $N(-d_2) = 0.5460$.

$$c_{\text{fut}} = e^{-0.042 \times 0.25}\,[7{,}850(0.4898) - 7{,}900(0.4540)] = 0.989555 \times (3{,}844.93 - 3{,}586.60) = 255.66 \text{ pts}$$

$$p_{\text{fut}} = e^{-0.042 \times 0.25}\,[7{,}900(0.5460) - 7{,}850(0.5102)] = 0.989555 \times (4{,}313.40 - 4{,}005.07) = 305.14 \text{ pts}$$

At A$25 per index point, one call contract costs $255.66 \times \$25 = \$6{,}391.50$.

Comparative Statics of Black-Scholes-Merton Inputs

Input Parameter Increase	European Call (c)	European Put (p)	American Call (C)	American Put (P)
Spot Price (S_0)	Increases (+)	Decreases (−)	Increases (+)	Decreases (−)
Strike Price (K)	Decreases (−)	Increases (+)	Decreases (−)	Increases (+)
Time to Expiry (T)	Increases (+) (usually)	Ambiguous (+/−)	Increases (+)	Increases (+)

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Input Parameter Increase	European Call (c)	European Put (p)	American Call (C)	American Put (P)
Volatility (σ)	Increases (+)	Increases (+)	Increases (+)	Increases (+)
Risk-Free Rate (r)	Increases (+)	Decreases (−)	Increases (+)	Decreases (−)
Dividend Yield / Cash Div (q, D)	Decreases (−)	Increases (+)	Decreases (−)	Increases (+)

4.3 Option Sensitivities (The Greeks) and Dynamic Delta Hedging

Derivatives trading desks manage option books by decomposing price exposure into partial derivatives known as **the Greeks**. Each Greek measures sensitivity to a specific BSM input parameter.

- **Delta ($\Delta = \frac{\partial f}{\partial S}$):** The rate of change of the option price with respect to a small change in the underlying asset price. For a European call on a continuous-yield asset, $\Delta_{\text{call}} = e^{-qT}N(d_1) \in (0, 1)$; for a put, $\Delta_{\text{put}} = e^{-qT}[N(d_1) - 1] \in (-1, 0)$. Delta approaches 1 (call) or -1 (put) deep in-the-money and 0 deep out-of-the-money.
- **Gamma ($\Gamma = \frac{\partial^2 f}{\partial S^2} = \frac{\partial \Delta}{\partial S}$):** The curvature (convexity) of the option value with respect to spot price, measuring how rapidly Delta changes as S moves. Gamma is identical for European calls and puts of the same strike and expiry, always positive for long options, and peaks sharply for near-expiry at-the-money options.
- **Theta ($\Theta = \frac{\partial f}{\partial t}$):** The rate of change of the option price with the passage of time (time decay), holding all else constant. Because options are wasting assets, Theta is normally negative for long calls and puts (except occasionally for deep ITM European puts).
- **Vega ($\mathrm{V} = \frac{\partial f}{\partial \sigma}$):** The sensitivity of the option price to a change in underlying volatility σ. Vega is identical and strictly positive for long European calls and puts, peaking for ATM options with longer maturities.
- **Rho ($\rho = \frac{\partial f}{\partial r}$):** The sensitivity of the option price to the risk-free interest rate r. Rising interest rates reduce the present value of paying the strike K, making call Rho positive and put Rho negative.

Greek	European Call Formula (with yield q)	European Put Formula (with yield q)	Sign (Long Position)
Delta (Δ)	$e^{-qT}N(d_1)$	$e^{-qT}[N(d_1) - 1]$	Call: +; Put: −
Gamma (Γ)	$\frac{e^{-qT}N'(d_1)}{S_0\sigma\sqrt{T}}$	$\frac{e^{-qT}N'(d_1)}{S_0\sigma\sqrt{T}}$	Both: +
Vega (V)	$S_0e^{-qT}N'(d_1)\sqrt{T}$	$S_0e^{-qT}N'(d_1)\sqrt{T}$	Both: +
Theta (Θ)	$-\frac{S_0e^{-qT}N'(d_1)\sigma}{2\sqrt{T}} + qS_0e^{-qT}N(d_1) - rKe^{-rT}N(d_2)$	$-\frac{S_0e^{-qT}N'(d_1)\sigma}{2\sqrt{T}} - qS_0e^{-qT}N(-d_1) + rKe^{-rT}N(-d_2)$	Normally − (time decay)
Rho (ρ)	$KTe^{-rT}N(d_2)$	$-KTe^{-rT}N(-d_2)$	Call: +; Put: −

Where: $N'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ is the standard normal probability density function. Using Worked Example 1's parameters ($S_0 = \$52, K = \$50, r = 4.2\%, q = 3.2\%, \sigma = 26\%, T = 0.50$), the call Greeks are $\Delta = 0.6202$, $\Gamma = 0.0389$, $\mathrm{V} = 13.6600$ (meaning a 1% volatility rise adds $\$0.1366$ to the call price), $\Theta = -\$3.6692/\text{yr}$ ($-\$0.0101/\text{day}$), and $\rho = 13.6862$ (a 1% rate rise adds $\$0.1369$).



The Gamma–Theta Trade-Off and Portfolio Neutralisation

Rewriting the BSM PDE for any delta-neutral portfolio ($\Delta_\Pi = 0$) on a non-dividend stock highlights a crucial relationship between Gamma and Theta:

$$\Theta_\Pi + \frac{1}{2}\sigma^2 S^2 \Gamma_\Pi = r\Pi$$

Where: Θ_Π, Γ_Π, and Π are the portfolio's Theta, Gamma, and market value. When Gamma is large and positive (long options), Theta is large and negative: convexity gains from large spot moves are paid for via steady daily time decay. Conversely, short-gamma option sellers earn positive daily Theta but suffer accelerating losses during sudden market jumps.

Because the underlying share has $\Delta = 1$ and $\Gamma = \mathrm{V} = 0$, trading the underlying stock alone can neutralise Delta, but cannot eliminate Gamma or Vega. To make a dealer portfolio simultaneously **Delta-, Gamma-, and Vega-neutral**, a trader must take positions w_1, w_2 in two distinct traded options with linearly independent Gamma and Vega sensitivities to solve the 2×2 linear system, and then trade shares to bring total portfolio Delta to zero.

Worked Example 4: Constructing a Delta–Gamma–Vega Neutral Dealer Portfolio

An Australian derivatives desk holds a delta-neutral OTC option book with current portfolio Greeks $\Delta_P = 0$, $\Gamma_P = -6{,}000$, and $\mathrm{V}_P = -9{,}000$. Two liquid ASX-traded options on the same underlying share are available: Option 1 has $\Delta_1 = 0.60$, $\Gamma_1 = 1.50$, $\mathrm{V}_1 = 2.00$; Option 2 has $\Delta_2 = 0.40$, $\Gamma_2 = 0.80$, $\mathrm{V}_2 = 1.60$. Calculate the exact quantities w_1, w_2 of Option 1 and Option 2, plus the underlying share trade w_S, required to achieve full Delta–Gamma–Vega neutrality.

1. **Set up the simultaneous equations for Gamma and Vega neutrality:**

$$1.50w_1 + 0.80w_2 = 6{,}000$$

$$2.00w_1 + 1.60w_2 = 9{,}000$$

2. **Solve the 2×2 linear system:** Multiply the first equation by 2: $3.00w_1 + 1.60w_2 = 12{,}000$. Subtracting the second equation gives:

$$(3.00 - 2.00)w_1 = 12{,}000 - 9{,}000 \implies w_1 = 3{,}000 \text{ (buy 3,000 Option 1)}$$

Substitute $w_1 = 3{,}000$ back into the second equation:

$$2.00(3{,}000) + 1.60w_2 = 9{,}000 \implies 1.60w_2 = 3{,}000 \implies w_2 = 1{,}875 \text{ (buy 1,875 Option 2)}$$

3. **Restore Delta neutrality using the underlying share:** Combined Delta after adding Options 1 and 2:

$$\Delta_{\text{new}} = 0 + 3{,}000(0.60) + 1{,}875(0.40) = 1{,}800 + 750 = +2{,}550$$

Therefore, the desk must **short 2,550 underlying shares** ($w_S = -2{,}550$) so that total portfolio Delta, Gamma, and Vega all equal zero.

Worked Example 5: Dynamic Delta Hedging Simulation Over Three Weekly Intervals

A dealer sells 100 European call option contracts (covering 10,000 shares) at $t = 0$ when $S_0 = \$50.00$ and $\Delta_0 = 0.5220$, borrowing at the risk-free rate $r = 5.00\%$ p.a. continuously compounded to finance the delta-hedging stock inventory. The portfolio is rebalanced weekly ($\Delta t = 1/52$ years, weekly interest factor $e^{0.05/52} - 1 = 0.000962$). Over the next three weeks, spot prices and corresponding BSM call Deltas evolve as $(S_1 = \$51.20, \Delta_1 = 0.6140)$, $(S_2 = \$50.40, \Delta_2 = 0.5480)$, and $(S_3 = \$52.10, \Delta_3 = 0.6950)$. Track the shares purchased/sold, weekly interest cost, and cumulative borrowing cost.

Week	Spot Price S_t (A$)	Call Delta Δ_t	Target Shares Held	Shares Traded (ΔQ)	Share Trade Cost (A$)	Weekly Interest (A$)	Cumulative Borrowing (A$)
0	50.00	0.5220	5,220	+5,220	+261,000.00	0.00	261,000.00
1	51.20	0.6140	6,140	+920	+47,104.00	251.08	308,355.08
2	50.40	0.5480	5,480	-660	-33,264.00	296.64	275,387.72
3	52.10	0.6950	6,950	+1,470	+76,587.00	264.92	352,239.64

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Notice the mechanical "buy high, sell low" cost incurred when dynamically hedging a short option position (negative Gamma): the dealer buys shares after prices rise (Week 1 and Week 3) and sells shares after prices fall (Week 2). In continuous time with constant volatility, the expected final cumulative cost equals the exact BSM option premium compounded to maturity.

Second-Order Taylor Series PnL Attribution and Cross-Greeks

Because option values are curved functions of the underlying spot price, a linear first-order Delta approximation understates gains on long-gamma books and understates losses on short-gamma books during large market shocks. Applying a second-order Taylor expansion in ΔS and first-order expansion in Δt produces the **Delta–Gamma–Theta PnL approximation**:

$$\Delta\Pi \approx \Delta \cdot \Delta S + \frac{1}{2}\Gamma(\Delta S)^2 + \Theta \cdot \Delta t$$

Where: $\Delta\Pi$ is the approximate change in portfolio value; ΔS is the discrete spot price shock; and Δt is the elapsed time in years. Consider the European call from Worked Example 1 ($S_0 = \$52.00$, $c_0 = \$4.8791$, $\Delta = 0.6202$, $\Gamma = 0.0389$, $\Theta = -\$3.6692/\text{yr}$). Suppose the stock drops sharply by $\Delta S = -\$4.00$ to $\$48.00$ over 5 calendar days ($\Delta t = 5/365$). A naive Delta-only estimate predicts $\Delta c \approx 0.6202(-4) = -\$2.4808$. Incorporating Gamma convexity and time decay yields:

$$\Delta c \approx 0.6202(-4) + \frac{1}{2}(0.0389)(-4)^2 + (-3.6692)\frac{5}{365} = -2.4808 + 0.3112 - 0.0503 = -\$2.2199$$

Repricing via the exact Merton BSM formula at $S = \$48.00, T = 0.50 - 5/365$ gives $c_{\text{new}} = \$2.6790$, an exact change of $-\$2.2001$. Adding the Gamma term cuts the approximation error from $\$0.2807$ down to less than $\$0.02$. Institutional exotic desks also monitor second-order cross-Greeks: **Vanna** ($\frac{\partial\Delta}{\partial\sigma} = \frac{\partial \mathrm{V}}{\partial S}$), **Volga/Vomma** ($\frac{\partial^2 f}{\partial\sigma^2}$), and **Charm** ($-\frac{\partial\Delta}{\partial t}$, delta bleed).

4.4 Volatility Estimation, Volatility Smiles and Australian Derivatives Risk Management

Volatility σ is the sole unobservable parameter in the Black-Scholes-Merton model. Practitioners distinguish between **historical (realised) volatility**, estimated backward from past market returns, and **implied volatility**, inferred forward from live option prices.

Historical Volatility, EWMA, and Implied Volatility

Given $n+1$ daily closing prices $S_0, S_1, \ldots, S_n$ (adjusted for dividends), define continuously compounded daily log returns as $u_i = \ln(S_i/S_{i-1})$. The sample daily standard deviation s and annualised historical volatility $\hat{\sigma}$ (assuming 252 ASX trading days per year) are:

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2}, \qquad \hat{\sigma} = s\sqrt{252}$$

Where: $u_i = \ln(S_i/S_{i-1})$ is the daily log return; $\bar{u}$ is the sample mean log return; and 252 is the standard annual trading-day scaling factor. Because financial volatility clusters in regimes, risk engines also apply **Exponentially Weighted Moving Average (EWMA)** models ($\sigma_n^2 = \lambda\sigma_{n-1}^2 + (1-\lambda)u_{n-1}^2$, typically $\lambda = 0.94$) and **GARCH(1,1)** models ($\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta\sigma_{n-1}^2$) to capture mean-reverting conditional variance.

Implied volatility (σ_{imp}) is the unique volatility input that equates the BSM theoretical formula to an observed market option price c_{mkt}. Because the BSM function cannot be inverted algebraically for σ, σ_{imp} is solved numerically via iterative root-finding such as the **Newton–Raphson method** using Vega (V) as the derivative: $\sigma_{k+1} = \sigma_k - \frac{c(\sigma_k) - c_{\text{mkt}}}{\mathrm{V}(\sigma_k)}$. In Australia, the **S&P/ASX 200 VIX (A-VIX)** aggregates implied volatilities across a strip of near-term XJO index options to provide a real-time 30-day forward-looking barometer of Australian equity market uncertainty.

Worked Example 6: Estimating Annualised Historical Volatility from ASX Daily Prices

An ASX-listed banking stock records six consecutive daily closing prices (in AUD): 100.00, 101.40, 99.80, 102.10, 103.50, and 101.90. Compute the five daily log returns, daily standard deviation, and annualised historical volatility assuming 252 trading days per year.

1. **Compute daily log returns $u_i = \ln(S_i/S_{i-1})$:**
   - $u_1 = \ln(101.40/100.00) = +0.013903$
   - $u_2 = \ln(99.80/101.40) = -0.015905$
   - $u_3 = \ln(102.10/99.80) = +0.022785$
   - $u_4 = \ln(103.50/102.10) = +0.013619$

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- $u_5 = \ln(101.90/103.50) = -0.015580$

2. **Compute sample mean and sample variance ($n = 5$):**

$$\bar{u} = \frac{0.013903 - 0.015905 + 0.022785 + 0.013619 - 0.015580}{5} = 0.003764$$

Sum of squared deviations $\sum_{i=1}^{5}(u_i - \bar{u})^2 = 0.00132277$.

$$s^2 = \frac{0.00132277}{5-1} = 0.00033069 \implies s = \sqrt{0.00033069} = 0.018185 \text{ (1.8185\% daily)}$$

3. **Annualise volatility:**

$$\hat{\sigma} = 0.018185 \times \sqrt{252} = 0.018185 \times 15.8745 = 0.2887 = 28.87\% \text{ p.a.}$$

GARCH(1,1) Conditional Variance Forecasting

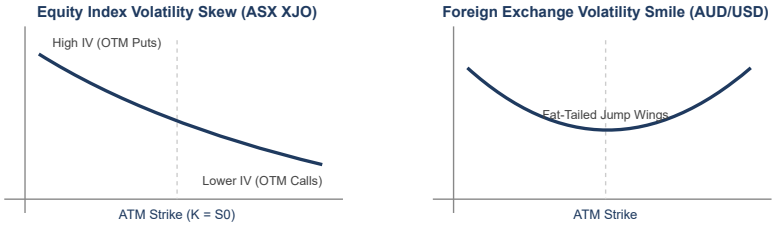
While sample historical volatility assigns equal weight to every observation in a rolling window, the **GARCH(1,1)** model updates conditional variance σ_n^2 dynamically using a long-run target variance $V_L = \frac{\omega}{1-\alpha-\beta}$, yesterday's squared return shock u_{n-1}^2, and yesterday's variance forecast σ_{n-1}^2:

$$\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2, \qquad \alpha + \beta < 1$$

Suppose an ASX risk model uses parameters $\omega = 0.000002$, $\alpha = 0.08$, and $\beta = 0.90$ (implying persistence $\alpha + \beta = 0.98$ and long-run daily variance $V_L = 0.000002/0.02 = 0.0001$, or 15.87% annualised). If yesterday's volatility estimate was $\sigma_{n-1} = 1.50\%$ daily and the market experiences a daily log return of $u_{n-1} = 2.50\%$, the updated daily variance forecast is $\sigma_n^2 = 0.000002 + 0.08(0.025)^2 + 0.90(0.015)^2 = 0.0002545$, corresponding to $\sigma_n = 1.5953\%$ daily or 25.32% annualised—demonstrating volatility clustering and gradual mean reversion back toward 15.87%.

Volatility Smiles, Equity Skew and Term Structure

If asset returns strictly followed constant-volatility Geometric Brownian Motion, all options on the same underlying and maturity would exhibit identical implied volatilities across every strike price K. In reality, plotting σ_{imp} against strike K reveals systematic empirical patterns called the **volatility smile** (in FX options, where deep OTM puts and calls both trade at higher implied volatility than ATM options due to fat-tailed jump risk) and the **volatility skew** (in equity index options such as XJO, where low-strike OTM puts trade at substantially higher implied volatility than high-strike OTM calls). Equity skew reflects both the *leverage effect* (falling equity prices raise corporate financial leverage and asset volatility) and institutional demand for downside crash protection ("crashophobia"). Practitioners also monitor the **volatility surface**, mapping implied volatility jointly across moneyness and time to maturity.



Value at Risk, Counterparty Credit Risk, and Australian Regulatory Governance

Enterprise risk management for derivative portfolios combines sensitivity limits (Greek limits) with portfolio-level tail-risk measures. **Value at Risk (VaR)** estimates the maximum loss expected over a holding period h at confidence level c (such as 10-day 99% VaR under Basel market-risk rules). While simple linear portfolios (forwards and swaps) can be evaluated with parametric **Delta-Normal VaR**, non-linear option books with material Gamma require **Historical Simulation** or **Monte Carlo Simulation** (or second-order Delta-Gamma approximations) to capture asymmetric convexity accurately. Beyond market risk, OTC derivatives create bilateral **Counterparty Credit Risk** quantified through **Credit Valuation Adjustment (CVA)**—the market value of expected default losses—and exacerbated by **Wrong-Way Risk** when exposure increases simultaneously as the counterparty's credit quality deteriorates.

Risk Methodology	Distributional Assumption	Suitability for Non-Linear Options
Linear Delta-Normal VaR	Multivariate normal returns; linear ΔS sensitivity.	Poor for high-Gamma option books; underestimates short-gamma tail losses.
Historical Simulation VaR	Non-parametric empirical return distribution (e.g., past 500 days).	Full revaluation captures Gamma, Vega, and fat tails, but limited by history.
Monte Carlo Simulation VaR	Simulated correlated stochastic paths (GBM, jump-diffusion, or GARCH).	Gold standard for complex option and CVA books; computationally intensive.
Expected Shortfall (ES)	Average loss conditional on exceeding the VaR threshold.	Coherent risk measure capturing tail severity; adopted under Basel FRTB / APS 116.

In Australia, derivatives governance, reporting, and accounting rest on four pillars:

- **ASIC Derivative Transaction Rules:** Enacted under Part 7.5A of the *Corporations Act 2001 (Cth)*, mandating trade reporting of OTC derivatives to ASIC-licensed trade repositories and central clearing of standardised G4 and AUD interest rate swaps for major financial institutions.
- **APRA Prudential Standards (CPS 226 & APS 116):** CPS 226 enforces mandatory variation margin and two-way initial margin (IM) exchange for non-centrally cleared derivatives among covered financial entities, while APS 116 governs regulatory capital charges for bank trading-book market risk.
- **AASB 9 Financial Instruments (Hedge Accounting):** Permits Australian corporations to align accounting profit recognition with economic risk management under fair value hedges, cash flow hedges, and net investment hedges when strict hedge documentation and effectiveness criteria are satisfied.
- **Governance Lessons from Derivative Failures:** High-profile failures—including Barings Bank (1995, unmonitored Nikkei 225 futures/straddle losses), Pasminco (2001, oversized AUD/USD FX hedge book), and National Australia Bank's FX options desk (2004, A$360 million unauthorised trading losses)—demonstrate that quantitative pricing models cannot substitute for strict segregation of front-office trading from middle-office risk control, independent mark-to-market valuation, and board-level risk appetite limits.

4.6 Advanced Applied Topics: Volatility Surfaces and Algorithmic Hedging in Australian Markets

Modern quantitative finance extends classical Black-Scholes-Merton assumptions to capture empirical market realities, most notably the phenomenon of the *volatility smile* and *volatility surface*. In liquid index options traded on the Australian Securities Exchange (such as S&P/ASX 200 Index options, ticker XJO), implied volatilities derived from market option prices systematically vary across different strike prices and expiration dates, directly violating the constant volatility assumption of the standard geometric Brownian motion framework.

The ASX 200 Volatility Skew Structure

For equity index options, market makers quote a pronounced *volatility skew* (or smirk), where out-of-the-money (OTM) put options trade at substantially higher implied volatilities than out-of-the-money call options. This structural asymmetry reflects market participants' collective demand for downside tail-risk portfolio insurance. Australian superannuation funds, institutional fund managers, and investment banks buy out-of-the-money puts to protect large diversified portfolios against catastrophic market crashes, bidding up the implied volatility of downside strikes.

Option Contract Type	Moneyness Ratio (K / S)	Typical Implied Volatility (IV)	Key Economic and Market Demand Driver
Deep Out-of-the-Money Put	0.85 (15% OTM)	24.5% to 28.0%	Institutional crash protection; tail risk hedging by superannuation funds.
At-the-Money Option (Call/Put)	1.00 (ATM)	14.0% to 16.5%	Baseline market pricing of expected near-term aggregate equity volatility.
Out-of-the-Money Call	1.10 (10% OTM)	11.5% to 13.0%	Covered call overwrite programs executed by yield-seeking institutional funds.

Worked Numerical Example: Volatility Interpolation and Greeks Calibration

Consider an option trader pricing a 6-month European option on the S&P/ASX 200 Index. The index currently trades at $S_0 = 7{,}800$ points, with a continuous risk-free rate of $r = 4.25\%$ per annum and an expected continuous dividend yield of $q = 3.80\%$ per annum.

A trader evaluates a 6-month strike $K = 7{,}400$ put option (OTM put) and a strike $K = 8{,}200$ call option (OTM call):

- At strike $K = 7{,}400$, the calibrated market implied volatility from the volatility surface is $\sigma_{\text{put}} = 19.5\%$.
- At strike $K = 8{,}200$, the calibrated market implied volatility is $\sigma_{\text{call}} = 13.2\%$.

- Time to expiration: $T = 0.5$ years.

Calculating the Black-Scholes-Merton d_1 parameter for the strike $K = 7{,}400$ put option:

$$d_1 = \frac{\ln(S_0/K) + (r - q + \sigma^2/2)T}{\sigma \sqrt{T}}$$

Where: $S_0 = 7{,}800$, $K = 7{,}400$, $r = 0.0425$, $q = 0.0380$, $\sigma = 0.195$, and $T = 0.5$.

Substituting the calibrated numerical inputs:

$$\ln(7{,}800/7{,}400) = \ln(1.054054) = 0.052643$$

$$(r - q + \sigma^2/2)T = (0.0425 - 0.0380 + 0.195^2/2)(0.5) = (0.0045 + 0.0190125)(0.5) = 0.011756$$

$$\sigma \sqrt{T} = 0.195 \times \sqrt{0.5} = 0.195 \times 0.707107 = 0.137886$$

$$d_1 = \frac{0.052643 + 0.011756}{0.137886} = \frac{0.064399}{0.137886} = 0.4670$$

$$d_2 = d_1 - \sigma \sqrt{T} = 0.4670 - 0.137886 = 0.3291$$

Using standard cumulative normal distribution probabilities $N(0.4670) \approx 0.6797$, $N(0.3291) \approx 0.6289$, and for the put option $N(-d_1) = 1 - 0.6797 = 0.3203$ and $N(-d_2) = 1 - 0.6289 = 0.3711$:

$$P_0 = Ke^{-rT}N(-d_2) - S_0e^{-qT}N(-d_1)$$

$$P_0 = 7{,}400e^{-0.0425\times0.5}(0.3711) - 7{,}800e^{-0.0380\times0.5}(0.3203)$$

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$$P_0 = 7{,}400(0.97897)(0.3711) - 7{,}800(0.98118)(0.3203) = 2{,}688.38 - 2{,}451.30 = 237.08 \text{ index points}$$

Multiplying by the ASX index contract multiplier ($10 per point), each contract is valued at $2,370.80 AUD. If the trader had naively utilized an unadjusted at-the-money volatility of 15.0% instead of the 19.5% skew-adjusted rate, the model would have substantially underpriced the protective put, leading to an inadequate hedging reserve and severe downside exposure during market stress.

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