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Bachelor of Commerce (B2001) | Level 2 • Part B Major | Monash University

Derivatives

Course Code: BFC2751 • Module 3 Notes

Module 3: Option Mechanics, Payoffs, Put-Call Parity & Binomial Option Pricing

3.1 Option Contracts, Market Conventions and Payoff Profiles

An **option** is a contingent-claim derivative that grants the holder (buyer) the right, without the obligation, to buy or sell an underlying asset at a predetermined **strike price** (or exercise price, K) on or before a specified **expiration date** (T). Because the option holder chooses to exercise only when doing so is financially advantageous, the holder's terminal payoff can never be negative. To acquire this asymmetric payoff right, the option buyer pays an upfront cash price—known as the **option premium**—to the option writer (seller) at contract initiation ($t = 0$).

Options are classified along two primary contractual dimensions. First, by transaction right: a **call option** gives the holder the right to *buy* the underlying asset at K, whereas a **put option** gives the holder the right to *sell* the underlying asset at K. Second, by exercise timing: a **European option** may be exercised only on the expiration date T, whereas an **American option** may be exercised on any trading day up to and including T. A third style, the **Bermudan option**, permits exercise only on a discrete set of predetermined dates prior to maturity.

ASX Exchange-Traded Options (ETOs) and Margining Conventions

In Australia, listed equity and index options trade on the **ASX Equity Derivatives Market** and clear through **ASX Clear**. Standard ASX single-stock Exchange-Traded Options (such as options on BHP Group, Commonwealth Bank, or CSL) typically carry a contract multiplier of **100 shares per contract** and follow American-style exercise, with physical delivery of shares upon exercise. By contrast, **S&P/ASX 200 Index Options (XJO)** follow European-style exercise, settle in cash against the index Opening Price Index Calculation (OPIC), and carry a multiplier of A$10 per index point.

When an ASX-listed company undertakes corporate actions such as bonus share issues, rights issues, or share splits, ASX Clear adjusts the option strike price and contract size to preserve economic equivalence for option holders. Unlike futures contracts, option buyers pay the full premium upfront in cash and face zero subsequent margin calls because their maximum possible loss is capped at the initial premium. Option writers (short calls and short puts), however, face potentially large or unlimited losses and must post initial and daily variation margin with ASX Clear under the Standard Portfolio Analysis of Risk (SPAN) margining methodology.

Terminal Payoffs, Profit Profiles, Intrinsic Value and Time Value

Let S_T denote the spot price of the underlying asset at maturity T, K the strike price, and c_0 or p_0 the initial option premium paid at $t = 0$. Ignoring the financing cost of the premium, the terminal payoffs and net profits at maturity across the four elementary option positions are:

$$\text{Long Call Payoff} = \max(S_T - K,\, 0), \qquad \text{Long Call Profit} = \max(S_T - K,\, 0) - c_0$$

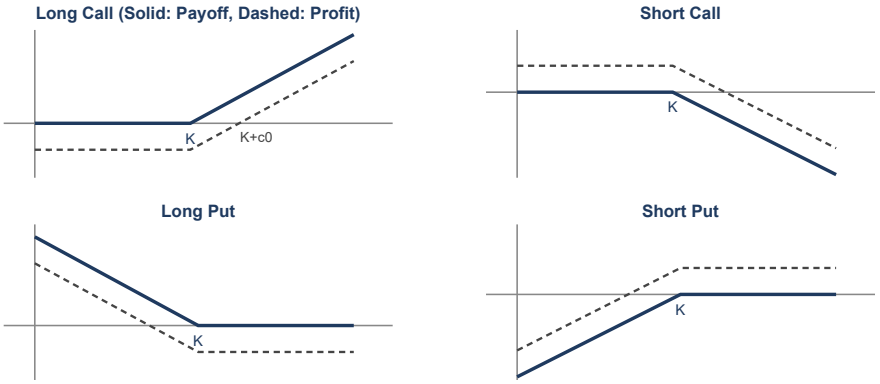
$$\text{Short Call Payoff} = -\max(S_T - K,\, 0), \qquad \text{Short Call Profit} = c_0 - \max(S_T - K,\, 0)$$

$$\text{Long Put Payoff} = \max(K - S_T,\, 0), \qquad \text{Long Put Profit} = \max(K - S_T,\, 0) - p_0$$

$$\text{Short Put Payoff} = -\max(K - S_T,\, 0), \qquad \text{Short Put Profit} = p_0 - \max(K - S_T,\, 0)$$

Where: S_T is the underlying asset price at expiry; K is the option strike price; c_0 is the initial call premium; and p_0 is the initial put premium.

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At any time $t < T$, an option's moneyness describes the relationship between current spot price S_t and strike K: an option is **in-the-money (ITM)** if immediate exercise would produce a positive cash inflow ($S_t > K$ for calls; $S_t < K$ for puts), **at-the-money (ATM)** if $S_t = K$, and **out-of-the-money (OTM)** if immediate exercise would yield zero or negative value ($S_t < K$ for calls; $S_t > K$ for puts). The total market premium decomposes into two components:

$$\text{Option Premium} = \text{Intrinsic Value} + \text{Time Value}$$

Where: **Intrinsic Value** is $\max(S_t - K, 0)$ for a call and $\max(K - S_t, 0)$ for a put; and **Time Value** (extrinsic value) equals the excess of market price over intrinsic value, reflecting the probability that favourable underlying price movements prior to expiry will increase the payoff.

Beyond standardised exchange-traded options, Australian financial markets feature several related option-like instruments: **warrants** (longer-dated options issued by financial institutions and listed on ASX), **convertible bonds** (corporate debt carrying an embedded call option to convert into ordinary shares), and **executive share options** (remuneration incentives valued under AASB 2 *Share-based Payment*).

ASX Clear Margining and Corporate Action Adjustments

When an ASX-listed company announces a corporate action, ASX Clear preserves the economic position of open ETO contracts under the ASX Clear Operating Rules. For a **bonus issue or share split** in ratio n-for-m, the strike price is scaled by m/n and the contract multiplier (or number of contracts) is scaled by n/m, keeping total exercise value unchanged. For **ordinary cash dividends**, no strike adjustment is made because expected dividends are already embedded in option premiums. However, for **special dividends** or capital returns exceeding ASX materiality thresholds, ASX Clear adjusts the strike price downward by the cash distribution per share on the ex-date. Similarly, writing uncovered ("naked") options requires posting initial margin equal to the option's mark-to-market liquidation value plus a risk-array buffer for adverse overnight price moves, whereas covered call writing (lodging the underlying ASX shares as specific cover with ASX Clear) reduces cash margin requirements substantially.

Feature	ASX Single-Stock ETOs (e.g., BHP, CBA)	S&P/ASX 200 Index Options (XJO)
Exercise Style	American (exercisable any trading day up to expiry)	European (exercisable only at expiration)
Contract Multiplier	100 underlying ordinary shares per contract	A$10 per index point
Settlement Method	Physical delivery of shares via CHESS	Cash-settled against Opening Price Index Calculation (OPIC)
Margining System	Buyer pays premium upfront; writer posts SPAN margin	Buyer pays premium upfront; writer posts SPAN margin

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3.2 Option Trading Strategies and Synthetic Positions

By combining options with different strikes, expiries, or underlying positions, investors engineer customised risk-return profiles tailored to directional views, income generation, or volatility forecasts.

Combining Options with the Underlying Asset

- **Protective Put (Portfolio Insurance):** Buying one share (S) together with one put option (p) at strike K. At expiry, the combined payoff is $\max(S_T, K) = S_T + \max(K - S_T, 0)$, establishing a hard floor K on portfolio value while preserving full upside appreciation.
- **Covered Call (Yield Enhancement):** Holding one share (S) while selling one call option (c) at strike $K > S_0$. Terminal payoff is $\min(S_T, K) = S_T - \max(S_T - K, 0)$. The writer collects upfront premium c_0 in exchange for capping upside gains at K.

Spreads and Combinations

A **spread** combines two or more options of the *same* type (all calls or all puts), whereas a **combination** mixes calls and puts on the same underlying asset:

Strategy	Construction (*K_1 < K_2 < K_3*)	Market Outlook	Maximum Profit / Maximum Loss
Bull Call Spread	Long 1 Call at K_1, Short 1 Call at K_2	Moderately bullish	Max Profit: $(K_2 - K_1) - \text{Net Cost}$; Max Loss: Net Premium Paid
Bear Put Spread	Long 1 Put at K_2, Short 1 Put at K_1	Moderately bearish	Max Profit: $(K_2 - K_1) - \text{Net Cost}$; Max Loss: Net Premium Paid
Long Butterfly Spread	Long 1 Call K_1, Short 2 Calls K_2, Long 1 Call K_3 (with $K_2 = \frac{K_1+K_3}{2}$)	Low volatility (price near K_2)	Max Profit: $(K_2 - K_1) - \text{Net Cost}$; Max Loss: Net Premium Paid
Long Straddle	Long 1 Call at K + Long 1 Put at K (same K, T)	High volatility (large move either way)	Max Profit: Unlimited upside; Max Loss: $c_0 + p_0$ (if $S_T = K$)
Long Strangle	Long 1 OTM Put at K_1 + Long 1 OTM Call at K_2	Very large move either way (cheaper than straddle)	Max Profit: Unlimited; Max Loss: $c_0 + p_0$ (for $K_1 \le S_T \le K_2$)
Protective Collar	Long Underlying + Long Put K_1 + Short Call K_2	Downside protection financed by capping upside	Terminal asset value bounded in $[K_1, K_2]$; often zero net premium

Zero-Cost Collars, Calendar Spreads and Box Spreads

Institutional treasurers and equity portfolio managers frequently construct multi-leg option structures to finance downside protection or isolate pure time-decay or interest-rate arbitrage:

- **Zero-Cost Protective Collar:** A hedger holding an underlying asset buys an OTM put at *K_1* < *S_0* and simultaneously sells an OTM call at *K_2* > *S_0* chosen so that $c(K_2) \approx p(K_1)$. The net upfront premium is approximately zero, locking the portfolio's terminal value within the corridor $[K_1, K_2]$.
- **Calendar (Time) Spread:** Selling a near-dated option and buying a longer-dated option at the same strike price K. Because short-dated ATM options experience faster time decay (higher Theta magnitude), the trader profits if the underlying stays near K until the near contract expires.
- **Box Spread Arbitrage:** Combining a bull call spread $(+c(K_1) - c(K_2))$ with a bear put spread $(+p(K_2) - p(K_1))$ of identical strikes and expiry guarantees a fixed terminal payoff of $K_2 - K_1$ regardless of S_T. Under no-arbitrage, its present value must equal $(K_2 - K_1)e^{-rT}$, making box spreads equivalent to synthetic risk-free lending or borrowing.

Worked Example 2: Constructing a Zero-Cost Equity Collar on an ASX Blue-Chip Holding

An Australian family office holds 10,000 shares of Commonwealth Bank of Australia (CBA) currently trading at $S_0 = \$120.00$ (portfolio value A$1,200,000). Seeking downside protection over the next 6 months without paying net cash premium, the manager enters a **zero-cost collar** by buying 100 six-month put option contracts (10,000 shares) at strike

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$K_1 = \$112.00$ costing $p = \$3.45$ per share and simultaneously writing 100 six-month call option contracts at strike $K_2 = \$130.00$ receiving $c = \$3.45$ per share. Evaluate the terminal portfolio value and net return across three terminal share prices: $S_T = \$102.00$, $S_T = \$122.00$, and $S_T = \$140.00$.

1. **Net Option Outlay at $t = 0$:** $10,000 \times (\$3.45 - \$3.45) = \$0.00$.
2. **Scenario A ($S_T = \$102.00$):** Shares fall to A$1,020,000. The long put at $K_1 = \$112.00$ is exercised, paying $10,000 \times (\$112.00 - \$102.00) = +\$100,000$. The short call expires worthless. Total hedged portfolio value = $\$1,020,000 + \$100,000 = \$1,120,000$ (maximum loss capped at -6.67% versus -15.00% unhedged).
3. **Scenario B ($S_T = \$122.00$):** Both options expire worthless. Total portfolio value equals the physical share value of $\$1,220,000$ ($+1.67\%$ gain).
4. **Scenario C ($S_T = \$140.00$):** Shares rise to A$1,400,000, while the short call is exercised against the portfolio for a payoff of $-10,000 \times (\$140.00 - \$130.00) = -\$100,000$. Total hedged portfolio value = $\$1,400,000 - \$100,000 = \$1,300,000$ (upside gain capped at $+8.33\%$).

Worked Example 3: Long Butterfly Spread Payoff and Profit Analysis

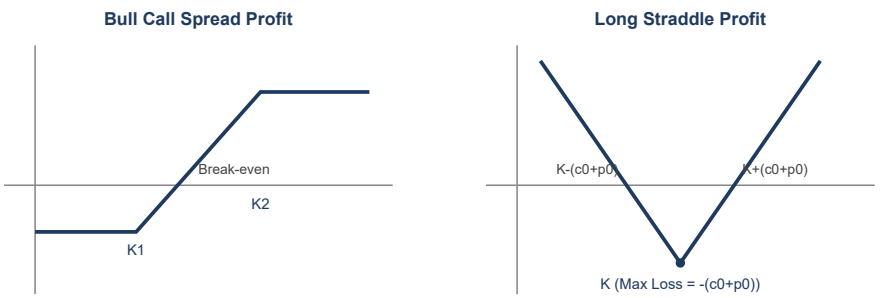
An options trader expects low volatility in an ASX-listed financial share currently trading at $S_0 = \$45.00$. The trader builds a 3-month **long call butterfly spread** by purchasing 1 call at $K_1 = \$40.00$ for $c_1 = \$6.20$, selling 2 calls at $K_2 = \$45.00$ for $c_2 = \$2.90$ each, and purchasing 1 call at $K_3 = \$50.00$ for $c_3 = \$1.10$ (per-share basis, scaled to 1,000 shares / 10 contracts). Determine the net premium, break-even prices, maximum profit, maximum loss, and complete payoff schedule.

1. **Net Initial Outlay:**

> $$\text{Net Cost per Share} = c_1 - 2c_2 + c_3 = \$6.20 - 2(\$2.90) + \$1.10 = \$1.50 \implies \$1,500.00 \text{ total}$$

2. **Maximum Profit, Maximum Loss, and Break-Even Points:** Maximum profit occurs at $S_T = K_2 = \$45.00$: $(K_2 - K_1) - \text{Net Cost} = \$5.00 - \$1.50 = \$3.50$/share (or A$3,500 total). Maximum loss is $-\$1.50$/share ($-\$1,500$ total) for $S_T \leq \$40$ or $S_T \geq \$50$. Break-even points are $K_1 + \$1.50 = \$41.50$ and $K_3 - \$1.50 = \$48.50$.

Terminal Price Region	Long $K_1 = \$40$	Short 2 $K_2 = \$45$	Long $K_3 = \$50$	Net Payoff / Share	Net Profit / Share
$S_T \leq \$40$	0	0	0	0	$-\$1.50$
$40 < S_T le $45	$S_T - 40$	0	0	$S_T - 40$	$S_T - 41.50$
$45 < S_T < $50	$S_T - 40$	$-2(S_T - 45)$	0	$50 - S_T$	$48.50 - S_T$
$S_T \geq \$50$	$S_T - 40$	$-2(S_T - 45)$	$S_T - 50$	0	$-\$1.50$



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Worked Example 1: Bull Call Spread Payoff and Profit Table on an ASX Equity

Shares of BHP Group trade at $S_0 = \$44.50$. A trader constructs a 3-month **bull call spread** by buying 10 call option contracts (covering 1,000 shares) at strike $K_1 = \$43.00$ for a premium of $c_1 = \$3.10$ per share, and simultaneously selling 10 call option contracts at strike $K_2 = \$47.00$ for $c_2 = \$1.20$ per share. Determine the net initial cost, break-even share price, maximum profit, maximum loss, and complete payoff/profit schedule across all terminal share price regions.

1. **Net Initial Premium Paid:**

> $$\text{Net Cost per Share} = c_1 - c_2 = \$3.10 - \$1.20 = \$1.90 \implies \text{Total Outlay} = 1,000 \times \$1.90 = \$1,900.00$$

2. **Break-Even Terminal Share Price:**

> $$S_T^* = K_1 + (c_1 - c_2) = \$43.00 + \$1.90 = \$44.90$$

3. **Maximum Profit and Maximum Loss (for 1,000 shares):**

> $$\text{Max Profit} = 1,000 \times [(K_2 - K_1) - (c_1 - c_2)] = 1,000 \times [(\$47.00 - \$43.00) - \$1.90] = \$2,100.00$$

> $$\text{Max Loss} = -\$1,900.00 \text{ (realised whenever } S_T \leq \$43.00)$$

Terminal Price Region	Long Call $K_1 = \$43$ Payoff	Short Call $K_2 = \$47$ Payoff	Total Payoff / Share	Net Profit / Share	Total Profit (1,000 Shares)
$S_T \leq \$43.00$	0	0	0	$-\$1.90$	$-\$1,900.00$
$43.00 < S_T < $47.00	$S_T - 43.00$	0	$S_T - 43.00$	$S_T - 44.90$	$1,000(S_T - 44.90)$
$S_T \geq \$47.00$	$S_T - 43.00$	$-(S_T - 47.00)$	\$4.00	+\$2.10	+\$2,100.00

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Institutional treasurers and equity portfolio managers frequently construct multi-leg option structures to finance downside protection or isolate pure time-decay or interest-rate arbitrage:

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Worked Example 3: Long Butterfly Spread Payoff and Profit Analysis

An options trader expects low volatility in an ASX-listed financial share currently trading at $S_0 = \$45.00$. The trader builds a 3-month **long call butterfly spread** by purchasing 1 call at $K_1 = \$40.00$ for $c_1 = \$6.20$, selling 2 calls at $K_2 = \$45.00$ for $c_2 = \$2.90$ each, and purchasing 1 call at $K_3 = \$50.00$ for $c_3 = \$1.10$ (per-share basis, scaled to 1,000 shares / 10 contracts). Determine the net premium, break-even prices, maximum profit, maximum loss, and complete payoff schedule.

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2. **Maximum Profit, Maximum Loss, and Break-Even Points:** Maximum profit occurs at $S_T = K_2 = \$45.00$: $(K_2 - K_1) - \text{Net Cost} = \$5.00 - \$1.50 = \$3.50$/share (or A$3,500 total). Maximum loss is $-\$1.50$/share ($-\$1,500$ total) for $S_T \leq \$40$ or $S_T \geq \$50$. Break-even points are $K_1 + \$1.50 = \$41.50$ and $K_3 - \$1.50 = \$48.50$.

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$40 < S_T le $45	$S_T - 40$	0	0	$S_T - 40$	$S_T - 41.50$
$45 < S_T < $50	$S_T - 40$	$-2(S_T - 45)$	0	$50 - S_T$	$48.50 - S_T$
$S_T \geq \$50$	$S_T - 40$	$-2(S_T - 45)$	$S_T - 50$	0	$-\$1.50$

3.3 Rational Pricing Bounds, Early Exercise & Put-Call Parity

Even before specifying a dynamic stochastic process for stock prices, strict no-arbitrage principles constrain the permissible values of European and American option prices. Let c, p denote European call and put prices, C, P denote American call and put prices, S_0 the current share price, K the strike price, r the continuously compounded risk-free rate, T the maturity, and D the present value of known dividends paid before T.

Upper and Lower No-Arbitrage Bounds

- **Upper Bounds:** A call option confers the right to buy one share, so it can never exceed the share price itself: $c \leq C \leq S_0$. A put option confers the right to sell one share for at most K, so an American put satisfies $P \leq K$ and a European put (paid only at T) satisfies $p \leq Ke^{-rT}$.
- **Lower Bounds (Non-Dividend-Paying Stock):**

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$$c \geq \max(0,\, S_0 - Ke^{-rT}), \qquad p \geq \max(0,\, Ke^{-rT} - S_0)$$

Where: $S_0 - Ke^{-rT}$ is proven by comparing Portfolio A (one European call plus cash Ke^{-rT}) against Portfolio B (one share): at T, Portfolio A pays $\max(S_T, K) \geq S_T$, so $c + Ke^{-rT} \geq S_0$. With known dividends of present value D, these bounds adjust to $c \geq \max(0, S_0 - D - Ke^{-rT})$ and $p \geq \max(0, D + Ke^{-rT} - S_0)$.

- **American Option Bounds:** Because an American option carries all rights of a European option plus early exercise flexibility, $C \geq c$ and $P \geq p$, and immediate exercise ensures $C \geq \max(0, S_0 - K)$ and $P \geq \max(0, K - S_0)$.

Optimal Early Exercise of American Options

A fundamental theorem of option pricing states that **it is never optimal to exercise an American call option on a non-dividend-paying stock prior to maturity**. Two economic forces drive this result: first, delaying payment of the strike K until T earns risk-free interest of $K(1 - e^{-r(T-t)}) > 0$ (provided $r > 0$); second, keeping the option alive preserves downside insurance if S_T subsequently falls below K. Mathematically, since $C \geq c \geq S_0 - K e^{-rT} > S_0 - K$ for $r > 0$, selling the call in the market always yields strictly more cash than exercising it early. Hence, for a non-dividend-paying stock, $C = c$.

When a stock pays a discrete dividend D_i at ex-dividend date t_i, however, the share price drops by approximately D_i at t_i, creating a reason to exercise an American call *immediately before* the ex-dividend instant if and only if the dividend captured exceeds the foregone interest on K plus lost insurance value—a necessary condition being $D_i > K(1 - e^{-r(T - t_i)})$. By contrast, **it can be optimal to exercise an American put option early even on a non-dividend-paying stock**. When a put is deeply in-the-money ($S_t \ll K$), the upside potential is bounded at K while waiting delays receipt of the cash strike K; exercising early allows immediate investment of K at the risk-free rate r, making $P > p$ in general.

Put-Call Parity

Consider two portfolios evaluated on a European option pair sharing the same underlying stock, strike K, and maturity T. **Portfolio A (Fiduciary Call)** holds one European call (c) plus zero-coupon risk-free cash of Ke^{-rT} (or $D + Ke^{-rT}$ if a dividend with present value D occurs before T). **Portfolio B (Protective Put)** holds one European put (p) plus one share (S_0). At expiration T, both portfolios deliver the exact same terminal value $\max(S_T, K)$ (after accounting for reinvested dividends). By the Law of One Price, their $t = 0$ values must be identical, establishing **Put-Call Parity**:

$$c + Ke^{-rT} = p + S_0 \quad \text{(Non-Dividend Stock)}, \qquad c + D + Ke^{-rT} = p + S_0 \quad \text{(Known Dividends PV} = D\text{)}$$

Where: c and p are European call and put premiums with identical strike K and maturity T; S_0 is the current spot share price; r is the continuously compounded risk-free rate; and D is the present value at $t = 0$ of discrete dividends paid during $(0, T]$. For a continuous dividend yield q, parity takes the form $c + Ke^{-rT} = p + S_0 e^{-qT}$.

Portfolio	Initial Outlay ($t = 0$)	Terminal Value if $S_T \leq K$	Terminal Value if $S_T > K$
Portfolio A (Fiduciary Call): 1 Call + Cash Ke^{-rT}	$c + Ke^{-rT}$	$0 + K = K$	$(S_T - K) + K = S_T$
Portfolio B (Protective Put): 1 Put + 1 Share	$p + S_0$	$(K - S_T) + S_T = K$	$0 + S_T = S_T$

Because American options permit early exercise, exact equality does not hold for American premiums C and P; instead, no-arbitrage imposes tight **American Put-Call Parity Inequalities**:

$$S_0 - D - K \leq C - P \leq S_0 - Ke^{-rT}$$

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Strike Monotonicity and Butterfly Convexity Conditions

Beyond Put-Call Parity across calls and puts of the same strike, no-arbitrage strictly governs how European option prices vary across different strikes *K_1 < K_2 < K_3* for a fixed maturity T. First, **monotonicity** requires that higher strikes reduce call prices ($c(K_1) \geq c(K_2)$) and raise put prices ($p(K_1) \leq p(K_2)$), with slopes bounded by discount factors: $0 \leq \frac{c(K_1)-c(K_2)}{K_2-K_1} \leq e^{-rT}$. Second, because a long butterfly spread can never produce a negative payoff at expiry, its initial cost must be non-negative, imposing **convexity in strike**:

$$c(K_2) \leq \lambda c(K_1) + (1-\lambda)c(K_3), \qquad \lambda = \frac{K_3 - K_2}{K_3 - K_1}$$

Where: $\lambda \in (0,1)$ weights the outer strikes K_1 and K_3 around the middle strike K_2. Violation of strike convexity implies a negative risk-neutral probability density in part of the terminal price distribution and permits a riskless static butterfly arbitrage.

Worked Example 4: Detecting and Exploiting a Put-Call Parity Arbitrage Violation

An ASX-listed non-dividend-paying stock currently trades at $S_0 = \$44.00$. Six-month ($T = 0.50$ years) European call and put options with strike $K = \$45.00$ trade at $c = \$3.20$ and $p = \$2.85$, respectively. The continuously compounded risk-free rate is $r = 4.00\%$ p.a. Test whether Put-Call Parity holds and, if an arbitrage exists, construct the exact riskless arbitrage portfolio on 1,000 shares.

1. **Test Put-Call Parity:**

$$Ke^{-rT} = \$45.00\, e^{-0.04\times 0.50} = \$45.00 \times 0.980199 = \$44.1089$$

$$\text{Fiduciary Call (LHS)} = c + Ke^{-rT} = \$3.20 + \$44.1089 = \$47.3089$$

$$\text{Protective Put (RHS)} = p + S_0 = \$2.85 + \$44.00 = \$46.8500$$

   Because *text{LHS} = $47.3089 > text{RHS} = $46.8500*, the call is overpriced relative to the put and stock by $\$0.4589$ per share.

2. **Execute the Arbitrage at $t = 0$ (for 1,000 shares):** Sell the expensive fiduciary call and buy the cheap protective put:
   - Short 10 call contracts (1,000 calls): receive $+1,000 \times \$3.20 = +\$3,200.00$.
   - Buy 10 put contracts (1,000 puts): pay $-1,000 \times \$2.85 = -\$2,850.00$.
   - Buy 1,000 shares in the spot market: pay $-1,000 \times \$44.00 = -\$44,000.00$.
   - Net cash outlay at $t = 0$ is $\$44,000.00 + \$2,850.00 - \$3,200.00 = \$43,650.00$, which is borrowed for 6 months at $r = 4.00\%$ p.a. Initial net cash flow = $\$0.00$.

3. **Verify Riskless Profit at $T = 0.50$ Years:** Loan repayment due at T equals $\$43,650.00\, e^{0.04\times 0.50} = \$44,531.82$.
   - **Case A (*S_T > $45.00*):** The short call is exercised against the arbitrageur, who delivers the 1,000 shares and receives $1,000 \times \$45.00 = \$45,000.00$; the long put expires worthless. Net profit = $\$45,000.00 - \$44,531.82 = +\$468.18$.
   - **Case B ($S_T \leq \$45.00$):** The short call expires worthless; the arbitrageur exercises the long put to sell the 1,000 shares for $\$45,000.00$. Net profit = $\$45,000.00 - \$44,531.82 = +\$468.18$.

   Notice that the terminal profit $\$468.18$ discounted back 6 months equals $\$468.18\, e^{-0.02} = \$458.91$, matching $1,000 \times \$0.4589$ exactly.

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3.4 Binomial Option Pricing: One-Step and Multi-Step CRR Trees

The **binomial option pricing model** (Cox, Ross, and Rubinstein, 1979) discretises the evolution of the underlying stock price into a recombining lattice over small time steps Δt. Over each step Δt, the stock price S moves either upward by factor $u > 1$ to Su or downward by factor $d < 1$ to Sd, with no-arbitrage requiring $d < e^{r\Delta t} < u$.

One-Step Binomial Model: Replicating Portfolio and Risk-Neutral Valuation

Consider a single period Δt where the stock begins at S_0 and transitions to either $S_u = S_0 u$ (option payoff f_u) or $S_d = S_0 d$ (option payoff f_d). To price the option without assuming investor risk preferences, construct a **riskless hedge portfolio** consisting of a long position in Δ shares and a short position in one option. Choosing Δ so that the portfolio attains the exact same value in both states ($S_u\Delta - f_u = S_d\Delta - f_d$) gives the hedge ratio:

$$\Delta = \frac{f_u - f_d}{S_0(u - d)}$$

Because this portfolio is completely riskless over Δt, it must earn the risk-free rate r to prevent arbitrage: $\Delta S_0 - f_0 = (\Delta S_0 u - f_u)e^{-r\Delta t}$. Solving for f_0 yields the **risk-neutral valuation formula**:

$$f_0 = e^{-r\Delta t}\left[pf_u + (1-p)f_d\right], \qquad p = \frac{e^{r\Delta t} - d}{u - d}$$

Where: f_0 is the current option value; f_u, f_d are the option values in the up and down states; u, d are the gross up and down stock return factors; r is the continuously compounded risk-free rate; Δt is the length of one time step in years; and p is the **risk-neutral probability** of an up move.

Crucially, the real-world physical probability of an up move and the stock's expected return μ cancel out completely in the pricing equation. Because options are priced by *replication* relative to the underlying stock, all investors—regardless of risk aversion—agree on the same option price as if they lived in a risk-neutral world where every asset earns the risk-free rate r.

Worked Example 5: One-Step Binomial Pricing via Replication and Risk-Neutral Valuation

An ASX stock trades at $S_0 = \$50.00$. Over the next 6 months ($\Delta t = 0.50$ years), it can either rise by 15% ($u = 1.15$, $S_u = \$57.50$) or fall by 12% ($d = 0.88$, $S_d = \$44.00$). The continuously compounded risk-free rate is $r = 5.00\%$ p.a. Value a 6-month European call option with strike $K = \$52.00$ using both the Replicating Portfolio method and the Risk-Neutral Valuation method.

1. **Terminal payoffs:** $f_u = \max(57.50 - 52.00, 0) = \$5.50$ and $f_d = \max(44.00 - 52.00, 0) = \$0.00$.
2. **Replicating Portfolio Method:**

$$\Delta = \frac{5.50 - 0.00}{57.50 - 44.00} = \frac{5.50}{13.50} = 0.407407 \text{ shares}$$

   Portfolio value at $t = 0.50$ in either state: $0.407407 \times 57.50 - 5.50 = \$17.9259$ (and $0.407407 \times 44.00 - 0 = \$17.9259$). Present value of riskless portfolio: $B = \$17.9259\, e^{-0.05\times0.50} = \$17.4833$. Therefore, $\Delta S_0 - c_0 = B \implies c_0 = 0.407407(50.00) - 17.4833 = \$20.3704 - \$17.4833 = \$2.8870$.
3. **Risk-Neutral Valuation Method:**

$$p = \frac{e^{0.05\times0.50} - 0.88}{1.15 - 0.88} = \frac{1.025315 - 0.88}{0.27} = \frac{0.145315}{0.27} = 0.538204$$

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$$c_0 = e^{-0.05\times0.50}\,[0.538204(5.50) + 0.461796(0)] = 0.975310 \times 2.960122 = \$2.8870$$

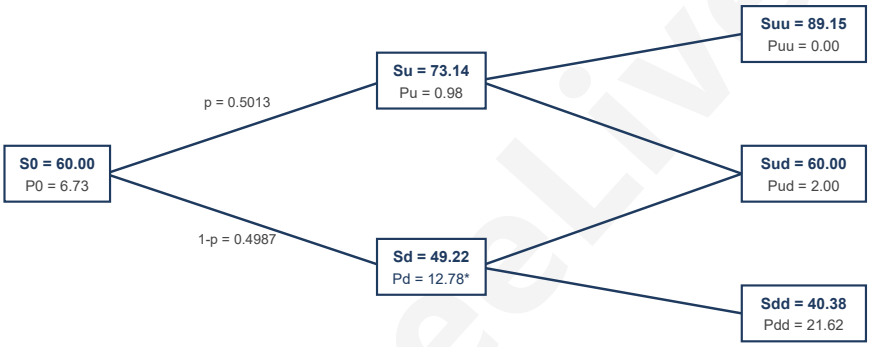
Both methods produce the exact call price of A$2.8870.

Multi-Step CRR Binomial Trees and American Early Exercise

To match a stock's observed annualised volatility σ across N time steps of length $\Delta t = T/N$, Cox, Ross, and Rubinstein (1979) parameterised the up and down factors as:

$$u = e^{\sigma\ \Delta t}, \qquad d = e^{-\sigma\ \Delta t} = \frac{1}{u}, \qquad p = \frac{e^{(r-q)\Delta t} - d}{u - d}$$

Where: σ is the annualised volatility of the underlying asset; $\Delta t = T/N$ is the step duration; and q is the continuous dividend yield (with $q = 0$ for non-dividend-paying stocks). Because $u \cdot d = 1$, an up-then-down move returns to the exact initial price ($S_0 ud = S_0$), ensuring a recombining tree with $i + 1$ stock nodes at step i.



Valuation proceeds by **backward induction** from maturity T back to $t = 0$. At maturity, node values equal terminal payoffs. At every earlier node (i,j), the continuation value is $f_{i,j}^{\text{cont}} = e^{-r\Delta t}[pf_{i+1,j+1} + (1-p)f_{i+1,j}]$. For a European option, $f_{i,j} = f_{i,j}^{\text{cont}}$. For an **American option**, the holder compares continuation against immediate exercise at every node:

$$P_{i,j}^{\text{American}} = \max(K - S_{i,j},\ e^{-r\Delta t}\,[pP_{i+1,j+1} + (1-p)P_{i+1,j}])$$

Worked Example 6: Two-Step CRR Binomial Valuation of an American Put Option

Consider a 1-year ($T = 1.0$) American put option on an ASX-listed non-dividend-paying share currently trading at $S_0 = \$60.00$ with strike $K = \$62.00$, annualised volatility $\sigma = 28\%$, and continuously compounded risk-free rate $r = 4.00\%$ p.a. Using a two-step CRR binomial tree ($\Delta t = 0.50$ years), compute the option price P_0 and identify any node where early exercise is optimal.

1. **Compute CRR tree parameters ($\Delta t = 0.50$):**

$$u = e^{0.28\ \ 0.50} = e^{0.197990} = 1.218950, \qquad d = \frac{1}{1.218950} = 0.820378$$

$$p = \frac{e^{0.04\times0.50} - 0.820378}{1.218950 - 0.820378} = \frac{1.020201 - 0.820378}{0.398572} = 0.501348, \qquad 1 - p = 0.498652$$

2. **Construct share prices across nodes:**
   - $t = 0.5$: $S_u = 60(1.218950) = \$73.1370$, $S_d = 60(0.820378) = \$49.2227$.
   - $t = 1.0$: $S_{uu} = 73.1370(1.218950) = \$89.1504$, $S_{ud} = \$60.0000$, $S_{dd} = 49.2227(0.820378) = \$40.3812$.
3. **Terminal payoffs at $t = 1.0$ ($\max(62 - S_T, 0)$):** $P_{uu} = \$0.0000$, $P_{ud} = \$62.00 - \$60.00 = \$2.0000$, $P_{dd} = \$62.00 - \$40.3812 = \$21.6188$.
4. **Backward induction to $t = 0.5$ (with early-exercise test):**
   - **Node u ($S_u = \$73.1370$):** Continuation = $e^{-0.02}[0.501348(0) + 0.498652(2.0000)] = 0.980199 \times 0.997304 = \$0.9776$. Intrinsic = $\max(62 - 73.1370, 0) = \$0.00$. Thus, $P_u = \max(0.9776, 0) = \$0.9776$ (hold).
   - **Node d ($S_d = \$49.2227$):** Continuation = $e^{-0.02}[0.501348(2.0000) + 0.498652(21.6188)] = 0.980199 \times 11.782949 = \$11.5496$. Intrinsic = $\max(62.00 - 49.2227, 0) = \$12.7773$. Because immediate exercise ($\$12.7773$) exceeds continuation ($\$11.5496$), **early exercise is optimal at node** d, so $P_d = \$12.7773$.
5. **Backward induction to $t = 0$:** Continuation = $e^{-0.02}[0.501348(0.9776) + 0.498652(12.7773)] = 0.980199 \times 6.861557 = \$6.7257$. Intrinsic at $t = 0$: $\max(62 - 60, 0) = \$2.0000$. Therefore, $P_0 = \max(6.7257, 2.0000) = \$6.7257$ (do not exercise at $t = 0$). For comparison, a European put on the exact same tree (substituting $P_d^{\text{Eur}} = \$11.5496$) would be worth $e^{-0.02}[0.501348(0.9776) + 0.498652(11.5496)] = \$6.1257$, showing an early-exercise premium of A$0.6000.

Extending Binomial Trees to Indices, Currencies and Futures; Convergence to BSM

The CRR binomial framework extends seamlessly to any underlying asset simply by adjusting the risk-neutral drift rate inside the up-move probability p while keeping $u = e^{\sigma \sqrt{\Delta t}}$, $d = 1/u$, and discounting at the domestic risk-free rate r:

Underlying Asset Class	Risk-Neutral Probability p	Economic Rationale
Non-Dividend Equity	$p = \frac{e^{r\Delta t} - d}{u - d}$	Expected total return equals domestic risk-free rate r.
Equity Index / Continuous Dividend Yield q	$p = \frac{e^{(r-q)\Delta t} - d}{u - d}$	Price appreciation equals $r - q$ after stripping dividend yield q.
Foreign Exchange Rate (S_0 domestic/foreign)	$p = \frac{e^{(r_d - r_f)\Delta t} - d}{u - d}$	Foreign risk-free rate r_f acts identically to a continuous yield q.
Futures Contract (F_0)	$p = \frac{1 - d}{u - d}$	Zero initial investment implies zero risk-neutral drift in futures prices.

In addition to pricing, the binomial tree supplies dynamic **hedge ratios (Δ)** at every node (i,j) via $\Delta_{i,j} = \frac{f_{i+1,j+1} - f_{i+1,j}}{S_{i+1,j+1} - S_{i+1,j}}$. In Worked Example 6 above, at $t = 0$ the American put's initial delta is $\Delta_0 = \frac{P_u - P_d}{S_u - S_d} = \frac{0.9776 - 12.7773}{73.1370 - 49.2227} = -0.4934$. As the number of time steps N increases (typically $N \geq 50$ in practice), the discrete CRR European option price converges smoothly—with mild even/odd oscillation around the exact strike placement—to the continuous-time Black-Scholes-Merton price, while American binomial lattices remain the industry workhorse for pricing early-exercise equity options and dividend-paying American contracts.

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